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Oscillations, Pseudorandomness and Hecke L-functions

by

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Abstract. We have given a study on the distribution of the convolution of the two arithmetic oscillating quantities $\mu(n)$ and $\{\alpha n\} - \frac{1}{2}$. In particular, we have been concerned with the uniform distribution of the Farey series whose denominators belong to the arithmetic progressions $b + an$ for given integers $a \geq 1$ and $1 \leq b \leq a$. We have shown that it has a connection with the Generalized Riemann Hypothesis (G.R.H.) for Dirichlet L-functions with Dirichlet characters mod a and a study on the Hecke L-functions connected with $\{\alpha n\} - \frac{1}{2}$, namely,

$$\sum_{n=1}^{\infty} \frac{\{\alpha n\} - \frac{1}{2}}{n^s}$$

and its extensions

$$\sum_{n=1}^{\infty} \frac{\{\alpha n\} - \frac{1}{2}}{n^s} \chi(n)$$

with Dirichlet characters χ, plays an important role. In Part I of this article, we are concerned with the distribution of the convolution of the two arithmetic oscillating quantities $\Lambda(n)$ and $\{\alpha n\} - \frac{1}{2}$, namely,

$$\sum_{\substack{dn \leq Q \\ dn \equiv b \,(\mathrm{mod}\, a)}} \Lambda(d) \left(\{\alpha n\} - \frac{1}{2} \right).$$

In particular, we shall apply an analytic study of the Hecke L-functions to the study of the weighted uniform distribution of the Farey series.

In Part II of this article, we shall study the arithmetic properties of the values of Hecke L-functions at $s = 1$. We shall show the transcendence results of the values for some cases. In the introduction of Part I, we shall also recall some of the arithmetic properties of the values of Hecke L-functions at $s = 0$, which seems to be of great interest from the other point of view.

In the subsequent paper, we shall study the distribution of the products

$$\mu(n) \cdot \left(\{\alpha n\} - \frac{1}{2} \right)$$

and also

$$\Lambda(n) \cdot \left(\{\alpha n\} - \frac{1}{2} \right).$$

We shall give an estimate on the sums

$$\sum_{\substack{n \le Q \\ n \equiv b \,(\mathrm{mod}\, a)}} \mu(n)\left(\{\alpha n\} - \frac{1}{2}\right)$$

and

$$\sum_{\substack{n \le Q \\ n \equiv b \,(\mathrm{mod}\, a)}} \Lambda(n)\left(\{\alpha n\} - \frac{1}{2}\right)$$

and give an analytic study of the zeta functions defined by

$$\sum_{\substack{n=1 \\ n \equiv b \,(\mathrm{mod}\, a)}}^{\infty} \frac{\mu(n)}{n^s}\left(\{\alpha n\} - \frac{1}{2}\right)$$

and

$$\sum_{\substack{n=1 \\ n \equiv b \,(\mathrm{mod}\, a)}}^{\infty} \frac{\Lambda(n)}{n^s}\left(\{\alpha n\} - \frac{1}{2}\right),$$

which are extensions of Davenport's sums

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\left(\{\alpha n\} - \frac{1}{2}\right)$$

and

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n}\left(\{\alpha n\} - \frac{1}{2}\right).$$

We shall at the same time give a study on the uniform distribution of the Farey series with prime denominators.

Table of Contents

Part I. Hecke L-functions and the Weighted Uniform Distribution of the Farey Series

§I-1. Introduction

Let $\{x\}$ be the fractional part of real x. Let α be a positive number. It is a classical result of Hardy-Littlewood [20] and Ostrowski that for a quadratic irrational α,

$$\sum_{n \le X} \left(\{\alpha n\} - \frac{1}{2} \right) = O(\log X) .$$

This estimate has been extended to any irrational α. Let ψ be a non-decreasing positive function defined at least for positive integers. An irrational number $\alpha (> 0)$ is said to be of type $< \psi$ (Cf. p. 121 of Kuipers and Niederreiter [26]), if

$$q \parallel q\alpha \parallel \ge \frac{1}{\psi(q)} \quad \text{for all integers} \quad q \ge 1 ,$$

where we put $\parallel x \parallel = \min(\{x\}, 1 - \{x\})$. It is well-known (cf. 3.5 on p.130 of Kuipers and Niederreiter [26]) that for any $\varepsilon_1 > 0$, almost all α are of type $< C(\alpha) \log^{1+\varepsilon_1}(2y)$, where $C(\alpha)$ is a positive constant that may depend on α. It is also well-known that for any algebraic irrational α, we can take

$$\psi(y) = C(\varepsilon) y^{\varepsilon}$$

for any $\varepsilon > 0$ with some positive constant $C(\varepsilon)$. The last result is a consequence of Thue-Siegel-Roth theorem (cf. p. 124 of Kuipers and Niederreiter [26]). Now Lang [27] showed that if α is irrational of type $< \psi$ and if $\frac{\psi(t)}{t}$ is decreasing, then

$$\sum_{n \le X} \left(\{\alpha n\} - \frac{1}{2} \right) \ll \int_1^X \frac{\psi(t)}{t} dt .$$

We have noticed a slight refinement in p. 124 of Fujii [19]: for any irrational α under the same assumption as above and for any real β,

$$\left| \sum_{n \le X} \left(\{\alpha n + \beta\} - \frac{1}{2} \right) \right| \le \frac{7}{2} \int_1^X \frac{\psi(t)}{t} dt .$$

The generating function of $\{\alpha n\} - \frac{1}{2}$ is the Hecke L-function $Z_\alpha(s)$ defined by

$$Z_\alpha(s) = \sum_{n=1}^{\infty} \frac{\{\alpha n\} - \frac{1}{2}}{n^s} .$$

When $D (\ge 1)$ is square free, $\equiv 2 \,\text{or}\, 3 \pmod 4$ and $\alpha = \sqrt{D}$ or $\frac{1}{\sqrt{D}}$, Hecke [21] gave a proof of the meromorphic continuation of $Z_\alpha(s)$ to the whole complex plane with simple poles at most at the points

$$s = -2k \pm 2\pi i \frac{m}{\log \eta_D} , \quad (k, m = 0, 1, 2, \dots)$$

and showed that for any positive ε,

$$\sum_{n\le X}\left(\{\alpha n\}-\frac{1}{2}\right)\log^2\frac{X}{n} = A_1\log^3 X + A_2\log^2 X + A_3\log X + \sum_{m=-\infty}^{\infty} C_m X^{\frac{2\pi i m}{\log \eta_D}} + O(X^{-1+\varepsilon}),$$

where A_1, A_2, A_3 and C_m are some constants, $C_m = O(|\ m\ |^{-2+\varepsilon})$ for $m \neq 0$ and η_D is the fundamental unit of the number field $\mathbf{Q}(\sqrt{D})$ or the square of it. Extending Hecke [21], Fujii [11] [12] has refined this result for the same α as follows. For any positive ε, we have

$$\sum_{n\le X}\left(\{\alpha n\}-\frac{1}{2}\right)\log\frac{X}{n} = \frac{1}{2}G_1(\alpha)\log^2 X + G_2(\alpha)\log X + \sum_{m=-\infty}^{\infty} C'_m X^{\frac{2\pi i m}{\log \eta_D}} + O(X^{-\frac{1}{3}+\varepsilon}),$$

where $G_1(\alpha)$ and $G_2(\alpha)$ are the Laurent coefficients in

$$\sum_{n=1}^{\infty}\frac{\{\alpha n\}-\frac{1}{2}}{n^s} = G_1(\alpha)\frac{1}{s} + G_2(\alpha) + \dots,$$

$C'_m = O(|\ m\ |^{-\frac{4}{3}+\varepsilon})$ for $m \neq 0$ and η_D is the same as above. Thus to get an asymptotic formula in such a form as

$$\sum_{n\le X}\left(\{\alpha n\}-\frac{1}{2}\right) \sim C\log X \quad (\text{as } X\to\infty)$$

seems to be of great interest, where C is some constant which may depend on a quadratic irrational α.

Concerning the evaluation of the Laurent coefficients, Hecke [21] obtained it as follows.

$$G_1(\sqrt{D}) = \begin{cases} 0 & \text{if } N(\varepsilon_D) = -1 \\ \dfrac{\zeta(1;v_1)\sqrt{D}}{\pi^2\log\varepsilon_D} & \text{if } N(\varepsilon_D) = 1 \end{cases}$$

and

$$G_2(\sqrt{D}) = \begin{cases} -\dfrac{\sqrt{D}}{12} + \dfrac{1}{8} & \text{if } N(\varepsilon_D) = -1 \\ \dfrac{\zeta(1;v_1)\sqrt{D}}{\pi^2\log\varepsilon_D}(\gamma_o + \log 2\pi) - \dfrac{\zeta'(1;v_1)\sqrt{D}}{2\pi^2\log\varepsilon_D} - \dfrac{\sqrt{D}}{12} + \dfrac{1}{8} & \text{if } N(\varepsilon_D) = 1, \end{cases}$$

where we put after Hecke

$$\zeta(s;v_1) = \sum_{(\mu)}\frac{\mathrm{sgn}(\mu\mu')}{|\ N(\mu)\ |^s},$$

(μ) runs over the non-zero principal integral ideals of $\mathbf{Q}(\sqrt{D})$, $N(\mu)$ is the norm of (μ), μ' is the conjugate of μ, $\mathrm{sgn}(\mu\mu')$ is the sign of $\mu\mu'$, γ_o is the Euler constant, ε_D is the fundamental unit of $\mathbf{Q}(\sqrt{D})$ and $\zeta'(s; v_1)$ is the derivative of $\zeta(s; v_1)$ (cf. p. 59 of Hecke [21]).

On the other hand, when a is an integer (≥ 1) and α is of the form

$$\alpha = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \ldots}}},$$

Hardy-Littlewood [20] gave a proof of the meromorphic continuation of $Z_\alpha(s)$ to the whole complex plane in a different way. Extending their method, Fujii [11] [12] has shown that for any quadratic irrational $\alpha > 0$, the values $G_1(\alpha)$ and $G_2(\alpha)$ can be evaluated explicitly in terms of the continued fraction expansion of α. For example, we have for any integer $k = 2, 3, 4, \ldots$

$$G_1\Big(\frac{\sqrt{k^2+4k}-k}{2}\Big) = \frac{1}{12}\frac{k-1}{\log\big(\frac{1}{2}(\sqrt{k^2+4k}+k+2)\big)}$$

and

$$G_2\Big(\frac{\sqrt{k^2+4k}-k}{2}\Big) = \frac{1}{\log B}\log\Big(\frac{\Gamma_2(B^2,(B,B^2-B))\rho_2((B-1,B))}{\rho_2((B,B^2-B))\Gamma_2(2B-1,(B-1,B))}\Big)$$
$$+\Big(\frac{1}{2}+\frac{1}{\log B}\log(B-B^{-1})\Big)\frac{k-1}{12}+\frac{1-k}{24k}(\sqrt{k^2+4k}-k)+\frac{1}{4},$$

where we put $B = \frac{1}{2}(\sqrt{k^2+4k}+k+2)$, Γ_2 is the Barnes double Γ-function (cf. Barnes [3][4] and Matsumoto [28]) and the positive constant $\rho_2((w_1, w_2))$ is defined by

$$-\log\rho_2((w_1,w_2)) = \lim_{a\to+0}\Big(\Big(\frac{\partial}{\partial s}\sum_{m,n\geq 1}(a+mw_1+nw_2)^{-s}\,|_{s=0}\Big)+\log a\Big).$$

When we take $k = 4n+2$, $n = 0, 1, 2, \ldots$ and $4n^2+8n+3$ is square free, then $\varepsilon_n = \sqrt{4n^2+8n+3}+2n+2$ is the fundamental unit of $\mathbf{Q}(\sqrt{4n^2+8n+3})$ and we have the following expression of $\zeta(1; v_1)$ and $\zeta'(1; v_1)$.

$$\zeta(1; v_1) = \frac{\pi^2}{12}\frac{4n+1}{\sqrt{4n^2+8n+3}}$$

and

$$\zeta'(1; v_1) = \frac{2\pi^2}{\sqrt{4n^2+8n+3}}\Big(-\log\Big(\frac{\Gamma_2(\varepsilon_n^2,(\varepsilon_n,\varepsilon_n^2-\varepsilon_n))\rho_2((\varepsilon_n-1,\varepsilon_n))}{\rho_2((\varepsilon_n,\varepsilon_n^2-\varepsilon_n))\Gamma_2(2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n))}\Big)\Big)$$
$$+\frac{4n+1}{12}\Big(\gamma_o+\log 2\pi-\log(\varepsilon_n-\varepsilon_n^{-1})-\frac{\log\varepsilon_n}{24}\Big(\frac{\sqrt{4n^2+8n+3}}{2n+1}+8n+5\Big)\Big).$$

Thus we see first that $G_1\big(\frac{\sqrt{k^2+4k}-k}{2}\big)$ is transcendental for any integer $k \geq 2$ and that $\zeta(1; v_1)$ for $\mathbf{Q}(\sqrt{4n^2+8n+3})$ is transcendental. However concerning the arithmetic nature of $G_2\big(\frac{\sqrt{k^2+4k}-k}{2}\big)$ and $\zeta'(1; v_1)$ for $\mathbf{Q}(\sqrt{4n^2+8n+3})$, our knowledge seems to be restricted. Concerning the values of $Z_\alpha(s)$ at $s = 1$, we shall give some problems or conjectures and some partial results in Part II below.

Let $\mu(n)$ be the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n \text{ is a product of } r\ (\geq 1) \text{ different prime numbers} \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known that the Riemann Hypothesis (R.H.) is equivalent to the statement that

$$\sum_{n \leq Q} \mu(n) \ll Q^{\frac{1}{2}+\varepsilon}$$

for any positive ε. Random behavior of the Möbius function is expressed as the Riemann Hypothesis. On the other hand, for any integer $n \geq 1$, we have

$$\mu(n) = \sum_{\substack{a=1 \\ (a,n)=1}}^{n} e^{2\pi\sqrt{-1}\frac{a}{n}}.$$

Consequently, the Riemann Hypothesis is equivalent to the statement that

$$\sum_{\substack{i=1 \\ f_i \in \mathcal{F}_Q}}^{A} e^{2\pi\sqrt{-1} f_i} \ll Q^{\frac{1}{2}+\varepsilon}$$

for any positive ε, where $\mathcal{F}_Q$ is the Farey series of the order $Q \geq 1$ which consists of the ascending sequence of the fractions

$$0 < f_1 < f_2 < f_3 < \cdots < f_{A-1} < f_A = 1$$

for which

$$f_i = \frac{a_i}{q_i},$$

$(a_i, q_i) = 1$ and $1 \leq a_i \leq q_i \leq Q$ and we put

$$A = A(Q) = \sum_{n \leq Q} \varphi(n)$$

with the Euler function $\varphi(n)$. We notice that

$$A(Q) = \frac{3}{\pi^2} Q^2 + O(Q(\log Q)^{\frac{2}{3}} (\log\log Q)^{\frac{4}{3}}) \quad \text{(cf. Walfisz [36])}.$$

Thus the Riemann Hypothesis is the statement concerning the uniform distribution of the Farey series. On the other hand, the uniform distribution of the Farey series $\mathcal{F}_Q$ can be seen from the following identity for any $0 < \alpha < 1$

$$\sum_{\substack{i=1, f_i \in \mathcal{F}_Q \\ f_i \leq \alpha}}^{A(Q)} 1 - \alpha A(Q) + \frac{1}{2} = -\sum_{dm \leq Q} \mu(d) \left(\{\alpha m\} - \frac{1}{2} \right).$$

This leads immediately to the asymptotic formula

$$\sum_{\substack{i=1, f_i \in \mathcal{F}_Q \\ f_i \leq \alpha}}^{A(Q)} 1 = \alpha A(Q) + O(Q \log Q)\,.$$

Moreover, the discrepancy of the uniform distribution of $\mathcal{F}_Q$ is described by the convolution of the two arithmetic oscillating quantities $\mu(n)$ and $\{\alpha n\} - \frac{1}{2}$ and the generating function is

$$-\frac{1}{\zeta(s)} Z_\alpha(s)\,,$$

where $\zeta(s)$ is the Riemann zeta function. We have given an analytic study of $Z_\alpha(s)$ in Fujii [11] [12] [19] and shown, in particular, that for almost all irrational α (including all algebraic irrational α) in $0 \leq \alpha \leq 1$ and for $\mid t \mid > t_o$

$$Z_\alpha\left(\frac{1}{2} + it\right) \ll \mid t \mid^{\frac{1}{6}+\varepsilon}\ .$$

Applying these, we have shown the following results in Fujii [19].

(i) Let $Q > Q_o$. For almost all irrational α (including all algebraic irrational α) in $0 \leq \alpha \leq 1$, we have

$$\sum_{\substack{i=1, f_i \in \mathcal{F}_Q \\ f_i \leq \alpha}}^{A(Q)} 1 - \alpha \cdot A(Q) = O(Q \cdot e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}})\,.$$

(i′) (On R.H.) Let $Q > Q_o$. For almost all irrational α (including all algebraic irrational α) in $0 \leq \alpha \leq 1$, we have

$$\sum_{\substack{i=1, f_i \in \mathcal{F}_Q \\ f_i \leq \alpha}}^{A(Q)} 1 - \alpha \cdot A(Q) = O(Q^{\frac{1}{2}+\varepsilon}) \quad \text{for any } \ \varepsilon > 0\,.$$

(ii) Let $Q > Q_o$. For any rational α in α in $0 \leq \alpha \leq 1$, we have

$$\sum_{\substack{i=1, f_i \in \mathcal{F}_Q \\ f_i \leq \alpha}}^{A(Q)} 1 - \alpha \cdot A(Q) = O(Q \cdot e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}})\,.$$

(ii′) (On R.H.) Let $Q > Q_o$. For any rational α in α in $0 \leq \alpha \leq 1$, we have

$$\sum_{\substack{i=1, f_i \in \mathcal{F}_Q \\ f_i \leq \alpha}}^{A(Q)} 1 - \alpha \cdot A(Q) = O(Q^{\frac{1}{2}+\varepsilon}) \quad \text{for any } \ \varepsilon > 0\,.$$

Thus we see that the expected value of the j-th fraction f_j in $\mathcal{F}_Q$ is $\frac{j}{A(Q)}$. In fact, the variance

$$\sum_{j=1}^{A(Q)} \left| \frac{j}{A(Q)} - f_j \right|^2$$

is connected with the Riemann Hypothesis. Franel-Landau's theorem (cf. pp. 263-267 of Edwards [8] and p. 283 of Titchmarsh [35]) tells us that the Riemann Hypothesis is equivalent to the statement that

$$\sum_{j=1}^{A(Q)} \left| \frac{j}{A(Q)} - f_j \right|^2 \ll Q^{-1+\varepsilon}$$

for any positive ε. Hence, on average, we must have

$$\sum_{\substack{i=1, f_i=\frac{a_i}{q_i}\le f_j \\ f_i\in\mathcal{F}_Q}}^{A(Q)} 1 - f_j A(Q) = O(Q^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$. Generally, we must have, on average,

$$\sum_{\substack{i=1, f_i\in\mathcal{F}_Q \\ f_i\le\alpha}}^{A(Q)} 1 - \alpha \cdot A(Q) = O(Q^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$. We should notice here that Dress [7] showed that

$$\sup_{0\le\alpha\le 1} \left| \frac{1}{A(Q)} \sum_{\substack{i=1, f_i\in\mathcal{F}_Q \\ f_i\le\alpha}}^{A(Q)} 1 - \alpha \right| = \frac{1}{Q}\,.$$

We turn our attention to the Generalized Riemann Hypothesis (G.R.H.) for Dirichlet L-functions $L(s,\chi)$ with Dirichlet characters χ mod a defined by

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}\,.$$

We notice that

$$\begin{aligned}\sum_{n\le Q} \mu(n)\chi(n) &= \sum_{n\le Q} \sum_{\substack{c=1 \\ (c,n)=1}}^{n} e^{2\pi\sqrt{-1}\frac{c}{n}} \chi(n) = \sum_{b=1}^{a} \chi(b) \sum_{\substack{n\le Q \\ n\equiv b\,(\mathrm{mod}\,a)}} \sum_{\substack{c=1 \\ (c,n)=1}}^{n} e^{2\pi\sqrt{-1}\frac{c}{n}} \\ &= \sum_{b=1}^{a} \chi(b) \sum_{\substack{i=1 \\ f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q, q_i\equiv b\,(\mathrm{mod}\,a)}}^{A(Q)} e^{2\pi\sqrt{-1} f_i}\end{aligned}$$

and

$$\sum_{\substack{i=1 \\ f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q, q_i\equiv b\,(\mathrm{mod}\,a)}}^{A(Q)} e^{2\pi\sqrt{-1}f_i} = \frac{1}{\varphi(a)}\sum_{\chi \bmod a}\bar{\chi}(b)\sum_{n\le Q}\mu(n)\chi(n)\,.$$

Hence, the statement that for any $\varepsilon > 0$ and for all b satisfying $(a, b) = 1$ and $1 \le b \le a$

$$\sum_{\substack{i=1 \\ f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q, q_i\equiv b\,(\mathrm{mod}\,a)}}^{A(Q)} e^{2\pi\sqrt{-1}f_i} = O(Q^{\frac{1}{2}+\varepsilon})$$

is equivalent to the statement that for any $\varepsilon > 0$ and for all Dirichlet character χ mod a

$$\sum_{n\le Q}\mu(n)\chi(n) = O(Q^{\frac{1}{2}+\varepsilon})\,.$$

Thus the Generalized Riemann Hypothesis for Dirichlet L-functions with Dirichlet characters mod a is the statement concerning the uniform distribution of the Farey sub-series $\mathcal{F}_Q(b, a)$ which consists of the reduced fractions in $\mathcal{F}_Q$ whose denominators belong to the arithmetic progressions $b + am$.

The uniform distribution of the Farey sub-series $\mathcal{F}_Q(b, a)$ can be seen from the following identity for $0 < \alpha < 1$ and for $(b, a) = 1$,

$$\begin{aligned}
&\sum_{\substack{i=1, f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q \\ f_i\le\alpha, q_i\equiv b\ (\bmod\ a)}}^{A(Q)} 1 - \alpha A(a, b, Q) + \frac{1}{2}\delta(b, 1) \\
&\quad = -\sum_{dm\le Q,\, dm\equiv b\ (\bmod\ a)} \mu(d)\left(\{\alpha m\} - \frac{1}{2}\right) \\
&\quad = -\frac{1}{\varphi(a)}\sum_{\chi \bmod a}\bar{\chi}(b)\sum_{dm\le Q}\mu(d)\chi(d)\left(\{\alpha m\} - \frac{1}{2}\right)\chi(m)\,,
\end{aligned}$$

where χ runs over all Dirichlet characters mod a and we put

$$A(a, b, Q) = \sum_{\substack{i=1, f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q \\ q_i\equiv b\ (\bmod\ a)}}^{A(Q)} 1$$

and

$$\delta(b, 1) = \begin{cases} 1 & \text{if } b \equiv 1\ (\bmod\ a) \\ 0 & \text{otherwise}\,. \end{cases}$$

We notice that for any positive integers a and b with $1 \le b \le a$, we have

$$A(a, b, Q) \equiv \sum_{n\le Q, n\equiv b\ (\bmod\ a)} \varphi(n)$$

$$= \frac{3}{\pi^2} \frac{1}{a \prod_{p|a} \left(1 - \frac{1}{p^2}\right)} \frac{\varphi(c)}{c} Q^2 + O\left(\sqrt{\frac{a}{c}} \log\left(\frac{a}{c} + 2\right) \cdot \left(\sum_{\delta|c} \frac{1}{\delta}\right) \cdot Q \log Q\right),$$

where we put $c = (a, b)$ and p runs over the prime numbers (cf. Lemma 1 on p. 111 of Fujii [19].) Thus the discrepancy of the uniform distribution of $\mathcal{F}_Q(b, a)$ is described by the zeta function

$$-\sum_{\chi \bmod a} \bar{\chi}(b) \frac{1}{L(s, \chi)} Z_\alpha(s, \chi),$$

where $Z_\alpha(s, \chi)$ is the Hecke L-function defined by

$$Z_\alpha(s, \chi) = \sum_{n=1}^{\infty} \frac{\{\alpha n\} - \frac{1}{2}}{n^s} \chi(n).$$

We (cf. Fujii [19]) have shown, as a special case, that for any Dirichlet character χ and for almost all irrational α (including all algebraic irrational α) in $0 \le \alpha \le 1$, $Z_\alpha(s, \chi)$ can be continued analytically to $\Re(s) > 0$ and

$$Z_\alpha\left(\frac{1}{2} + it, \chi\right) \ll | t |^{\frac{1}{6}+\varepsilon} .$$

Applying these, we have shown the following results in Fujii [19], where we have treated a more general case for $1 \le b \le a$.

(iii) Let $Q > Q_o$. Let a and b be integers satisfying $1 \le b \le a$. For almost all irrational α (including all algebraic irrational α) in $0 \le \alpha \le 1$, we have

$$\sum_{\substack{i=1, f_i=\frac{a_i}{q_i} \in \mathcal{F}_Q \\ f_i \le \alpha, q_i \equiv b \ (\mathrm{mod}\ a)}}^{A(Q)} 1 - \alpha A(a, b, Q) = O(Q \cdot e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}}).$$

(iii′) (On G.R.H.) Let $Q > Q_o$. Let a and b be integers satisfying $1 \le b \le a$. For almost all irrational α (including all algebraic irrational α) in $0 \le \alpha \le 1$, we have

$$\sum_{\substack{i=1, f_i=\frac{a_i}{q_i} \in \mathcal{F}_Q \\ f_i \le \alpha, q_i \equiv b \ (\mathrm{mod}\ a)}}^{A(Q)} 1 - \alpha A(a, b, Q) = O(Q^{\frac{1}{2}+\varepsilon}) \quad \text{for any } \varepsilon > 0$$

For rational α, the situation is different from the case for the Farey series $\mathcal{F}_Q$ and should be compared with (ii) stated above. Namely, the second main term may appear as follows.

(iv) Suppose that $Q > Q_o$. Let a and b be integers satisfying $1 \le b \le a$. For any rational α in $0 \le \alpha \le 1$, we have

$$\sum_{\substack{i=1, f_i=\frac{a_i}{q_i} \in \mathcal{F}_Q \\ f_i \le \alpha, q_i \equiv b \ (\mathrm{mod}\ a)}}^{A(Q)} 1 - \alpha A(a, b, Q) = C(\alpha, a, b) \cdot Q + O(Q \cdot e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}}),$$

where we put, for $\alpha = \frac{p}{q}$ with $(p,q) = 1$,

$$C(\alpha, a, b) = -\frac{1}{c\varphi(\frac{a}{c})} \sum_{\delta | c} \mu(\delta) \frac{1}{[\frac{q}{d}, \frac{a}{c}]} \sum_{\chi \bmod \frac{a}{c}, \chi \neq \chi_{o,\frac{a}{c}}} \bar{\chi}(\frac{b}{c}) \frac{1}{L(1, \chi\chi_{o,a})}$$

$$\cdot \sum_{h=1}^{\frac{q}{d}} (\{\frac{\frac{c}{\delta d} p}{q/d} h\} - \frac{1}{2}) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\ (\frac{q}{d}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j),$$

χ running over all Dirichlet characters mod $\frac{a}{c}$, $\chi_{o,\frac{a}{c}}$ and $\chi_{o,a}$ being the principal characters mod $\frac{a}{c}$ and a, respectively, $c = (a, b)$, $d = (\frac{c}{\delta}, q)$ for $\delta \mid c$ and $[\frac{q}{d}, \frac{a}{c}]$ is the least common multiple of $\frac{q}{d}$ and $\frac{a}{c}$.

(iv') (On G.R.H.) Suppose that $Q > Q_o$. Let a and b be integers satisfying $1 \leq b \leq a$. For any rational α in $0 \leq \alpha \leq 1$, we have for any $\varepsilon > 0$

$$\sum_{\substack{i=1, f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q \\ f_i \leq \alpha, q_i \equiv b \ (\mathrm{mod}\ a)}}^{A(Q)} 1 - \alpha A(a, b, Q) = C(\alpha, a, b) \cdot Q + O(Q^{\frac{1}{2}+\varepsilon}) \quad \text{for any } \varepsilon > 0$$

For convenience, we recall some example of the values of the constants $C(\alpha, a, b)$.

EXAMPLE. For any reduced fraction $0 < \frac{p}{q} \leq 1$ and for $b = 1, 2$ and 3, we have

$$\lim_{Q \to \infty} \frac{1}{Q} \left(\sum_{\substack{i=1, f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q \\ f_i \leq \frac{p}{q}, q_i \equiv b \ (\mathrm{mod}\ 3)}}^{A(Q)} 1 - \frac{p}{q} A(3, b, Q) \right)$$

$$= \begin{cases} 0 & \text{if } 3 \not| q \text{ or } b = 3 \\ \frac{\sqrt{3}}{2\pi q} & \text{if } 3 \mid q \text{ and } bp \equiv 1 \ (\mathrm{mod}\ 3) \\ -\frac{\sqrt{3}}{2\pi q} & \text{if } 3 \mid q \text{ and } bp \equiv 2 \ (\mathrm{mod}\ 3)\,. \end{cases}$$

We may expect that for most of f_j, $1 \leq j \leq A(Q)$,

$$\frac{j(a, b, Q)}{A(a, b, Q)} - f_j$$

is small, where we put

$$j(a, b, Q) = \sum_{\substack{i=1, f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q \\ q_i \equiv b \ (\mathrm{mod}\ a)}}^{j} 1$$

In fact, the variance is connected with the Generalized Riemann Hypothesis. We have shown the following which gives an extension of Franel-Landau's theorem for $\mathcal{F}_Q$ to the Farey sub-series $\mathcal{F}_Q(a, b)$.

(v) Let a be an integer ≥ 1. Let $Q > Q_o$ be an integer. Then the Generalized Riemann Hypothesis for all Dirichlet L-function $L(s, \chi)$ with Dirichlet characters χ mod a is equivalent to the statement that

$$\sum_{\substack{j=1 \\ f_j \in \mathcal{F}_Q}}^{A(Q)} \left| \frac{j(a, b, Q)}{A(a, b, Q)} - f_j \right|^2 \ll Q^{-1+\varepsilon}$$

holds for all integer b in $1 \leq b \leq a$ satisfying $(a, b) = 1$ and for any positive ε, where the constant involved in $\ll$ may depend on ε, a and b.

We now proceed to describe the problem which we shall treat in this article. Let $\Lambda(n)$ be the von-Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime number } p \text{ and an integer } k \geq 1 \\ 0 & \text{otherwise}. \end{cases}$$

It is well known that the Riemann Hypothesis is equivalent to the statement that for any $\varepsilon > 0$,

$$\sum_{n \leq Q} \Lambda(n) - Q = O(Q^{\frac{1}{2}+\varepsilon}).$$

Since

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d},$$

we have

$$\begin{aligned} \sum_{n \leq Q} \Lambda(n) &= \sum_{n \leq Q} \sum_{d|n} \mu(d) \log \frac{n}{d} = \sum_{n \leq Q} \mu(n) \sum_{m \leq \frac{Q}{n}} \log m \\ &= \sum_{n \leq Q} \sum_{\substack{a=1 \\ (a,n)=1}}^{n} e^{2\pi\sqrt{-1}\frac{a}{n}} \left(\sum_{m \leq \frac{Q}{n}} \log m \right) = \sum_{\substack{i=1 \\ f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q}}^{A} e^{2\pi\sqrt{-1} f_i} \left(\sum_{m \leq \frac{Q}{q_i}} \log m \right). \end{aligned}$$

It is not clear at this point how the uniform distribution of $f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q$ with the weight $\sum_{m \leq \frac{Q}{q_i}} \log m$ is connected with the Riemann Hypothesis. On the other hand, we can look at the uniform distribution of f_i with the weight $\sum_{m \leq \frac{Q}{q_i}} \log m$ directly as follows. The details for a more general case will be presented in the section 2 below.

$$\begin{aligned} \sum_{\substack{i=1, f_i = \frac{a_i}{q_i} \in F_Q \\ f_i \leq \alpha}}^{A} \left(\sum_{m \leq \frac{Q}{q_i}} \log m \right) &= \sum_{n \leq Q} \sum_{h \leq \alpha n} \log(h, n) \\ &= \sum_{n \leq Q} \sum_{h \leq \alpha n} \sum_{d|(h,n)} \Lambda(d) = \sum_{d \leq Q} \Lambda(d) \sum_{m \leq \frac{Q}{d}} [\alpha m] \\ &= \alpha \sum_{d \leq Q} \Lambda(d) \sum_{m \leq \frac{Q}{d}} m - \sum_{d \leq Q} \Lambda(d) \sum_{m \leq \frac{Q}{d}} \left(\{\alpha m\} - \frac{1}{2} \right) - \frac{1}{2} \sum_{n \leq Q} \log n, \end{aligned}$$

where $x - [x] = \{x\}$ for a real x. Since

$$\sum_{d \le Q} \Lambda(d) \sum_{m \le \frac{Q}{d}} \left(\{\alpha m\} - \frac{1}{2} \right) \ll Q \log Q$$

and

$$\sum_{n \le Q} \log n \ll Q \log Q \, ,$$

we get

$$\sum_{\substack{i=1,\, f_i = \frac{a_i}{q_i} \in F_Q \\ f_i \le \alpha}}^{A} \left(\sum_{m \le \frac{Q}{q_i}} \log m \right) = \alpha \cdot \sum_{i=1,\, f_i = \frac{a_i}{q_i} \in F_Q}^{A} \left(\sum_{m \le \frac{Q}{q_i}} \log m \right) + O(Q \log Q) \, .$$

Namely, we get the uniform distribution of f_i with the weight $\sum_{m \le \frac{Q}{q_i}} \log m$. We may expect that

$$\sum_{\substack{i=1 \\ f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q}}^{j} \left(\sum_{m \le \frac{Q}{q_i}} \log m \right) - f_j \sum_{\substack{i=1 \\ f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q}}^{A} \left(\sum_{m \le \frac{Q}{q_i}} \log m \right)$$

is small for most of $1 \le j \le A(Q)$. In fact, the variance

$$\sum_{\substack{j=1 \\ f_j \in \mathcal{F}_Q}}^{A} \left| \sum_{\substack{i=1 \\ f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q}}^{j} \left(\sum_{m \le \frac{Q}{q_i}} \log m \right) - f_j \sum_{\substack{i=1 \\ f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q}}^{A} \left(\sum_{m \le \frac{Q}{q_i}} \log m \right) \right|^2$$

is connected with the Riemann Hypopthsis. More precisely, we have given a restatement of the Riemann Hypothesis in terms of the Farey series as follows (cf. pp. 198–200 of Fujii [9]).

(vi) The Riemann Hypothesis is equivalent to the statement that

$$\sum_{\substack{j=1 \\ f_j \in \mathcal{F}_Q}}^{A} \Bigg| \sum_{\substack{i=1 \\ f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q}}^{j} \left(\sum_{m \le \frac{Q}{q_i}} \log m \right) - f_j \sum_{\substack{i=1 \\ f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q}}^{A} \left(\sum_{m \le \frac{Q}{q_i}} \log m \right)$$
$$+ \frac{1}{2} \sum_{m \le Q} \log m + Q \sum_{d \le Q} \frac{\{f_j n\} - \frac{1}{2}}{n} \Bigg|^2 \ll Q^{3+\varepsilon}$$

for any positive ε.

The first problem which we shall treat in the present article is to extend (vi) to the Generalized Riemann Hypothesis. In fact, we shall show that the uniform distribution of the Farey series $f_i = \frac{a_i}{q_i} \in \mathcal{F}_Q$ with the weight

$$\sum_{\substack{m \le \frac{Q}{q_i} \\ m q_i \equiv b \, (\mathrm{mod}\, a)}} \log m$$

is connected with the Generalized Riemann Hypothesis. To state our results we put

$$F_\alpha(a,b,Q)=\sum_{\substack{i=1,\,f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q\\ f_i\le\alpha}}^{A(Q)}\Bigg(\sum_{\substack{m\le\frac{Q}{q_i}\\ mq_i\equiv b\,(\mathrm{mod}\,a)}}\log m\Bigg)$$

and

$$F(\alpha;a,b,Q)=F_\alpha(a,b,Q)-\alpha F_1(a,b,Q)+\frac{1}{2}\sum_{\substack{n\le Q\\ n\equiv b\,(\mathrm{mod}\,a)}}\log n$$
$$+\frac{Q}{\varphi(a)}\sum_{\substack{m\le Q\\ (m,a)=1}}\frac{\{\alpha m\}-\frac{1}{2}}{m}\,.$$

We denote $F_1(a,b,Q)$ by $F(a,b,Q)$. On the size of $F(a,b,Q)$ we notice the following lemma which will be proved in the section 3.

LEMMA 1. *Let $Q>Q_o$. For any positive integers a and b with $1\le b\le a$, we have*

$$\begin{aligned}F(a,b,Q)\equiv&\sum_{i=1,\,f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q}^{A}\Bigg(\sum_{\substack{m\le\frac{Q}{q_i}\\ mq_i\equiv b\,(mod\,a)}}\log m\Bigg)\\
=&\frac{Q^2}{2a}\Bigg(-\frac{6}{\pi^2 a\prod_{p|a}\left(1-\frac{1}{p^2}\right)}L'(2,\chi_{o,\frac{a}{c}})\\
&+(1-\delta_1(c))\sum_{k=1}^{r}\frac{\log p_k}{p_k^{\nu_k}}\Bigg(\frac{p_k-p_k^{\nu_k}}{1-p_k}+\frac{p_k^2}{p_k^2-\chi_{o,\frac{a}{p_k^{\nu_k}}}(p_k)}\Bigg)\Bigg)\\
&+O\Bigg(Q\sqrt{\frac{a}{c}}\log\left(\frac{a}{c}+2\right)(\log Q+\log c)\Bigg)\\
&+O\Bigg(\frac{Q}{a}\Bigg(1+(1-\delta_1(c))\frac{1}{Q}\sum_{k=1}^{r}p_k^{\nu_k}\log p_k\Bigg)\Bigg)\,,\end{aligned}$$

where we put $c=(a,b)$ and if $c>1$, we write $c=\prod_{k=1}^{r}p_k^{\nu_k}$ with prime numbers p_k and integers $\nu_k\ge1$, $\chi_{o,\frac{a}{c}}$ and $\chi_{o,\frac{a}{p_k^{\nu_k}}}$ are the principal characters mod $\frac{a}{c}$ and $\frac{a}{p_k^{\nu_k}}$, respectively, p runs over the prime numbers and we put

$$\delta_1(c)=\begin{cases}1 & \textit{if }\ c=1\\ 0 & \textit{if }\ c>1\,.\end{cases}$$

Now our extension of (vi) to the Generalized Riemann Hypothesis may be stated as follows.

THEOREM I-1. *Let $Q > Q_o$. The Generalized Riemann Hypothesis for all $L(s, \chi)$ with Dirichlet characters mod a is equivalent to the statement that*

$$\int_0^1 | F(\alpha; a, b, Q) |^2 \, d\alpha \ll Q^{1+\varepsilon}$$

for any $\varepsilon > 0$ and for any b satisfying $(a, b) = 1$ and $1 \le b \le a$.

THEOREM I-2. *Let $Q > Q_o$. The Generalized Riemann Hypothesis for all $L(s, \chi)$ with Dirichlet characters mod a is equivalent to the statement that*

$$\sum_{\substack{j=1\\ f_j\in\mathcal{F}_Q}}^{A} \Bigg| \sum_{i=1,\, f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q}^{j} \Bigg(\sum_{\substack{m\le\frac{Q}{q_i}\\ mq_i\equiv b\,(\mathrm{mod}\,a)}} \log m \Bigg) - f_j \sum_{\substack{i=1,\, f_i=\frac{a_i}{q_i}\in\mathcal{F}_Q\\ f_i\le\alpha}}^{A} \Bigg(\sum_{\substack{m\le\frac{Q}{q_i}\\ mq_i\equiv b\,(\mathrm{mod}\,a)}} \log m \Bigg) + \frac{1}{2} \sum_{\substack{n\le Q\\ n\equiv b\,(\mathrm{mod}\,a)}} \log n + \frac{Q}{\varphi(a)} \sum_{\substack{m\le Q\\ (m,a)=1}} \frac{\{f_j m\} - \frac{1}{2}}{m} \Bigg|^2 \ll Q^{3+\varepsilon}$$

for any $\varepsilon > 0$ and for any b satisfying $(a, b) = 1$ and $1 \le b \le a$.

The second problem which we shall consider in this article is to get analogues of (i), (i′), (ii), (ii′), (iii), (iii′), (iv) and (iv′) stated above. We shall prove the following theorems.

THEOREM I-3. *Let $Q > Q_o$. Let α be irrational of type $< \psi$, where ψ satisfies $\psi(y) \ll y^{\nu_o}$ for some ν_o in $0 \le \nu_o < 1$ and $0 < \alpha < 1$. Then we have, for any $1 \le b \le a$,*

$$F_\alpha(a, b, Q) - \alpha F(a, b, Q) = -\frac{1}{2a}(Q \log Q - Q) - \frac{1}{\varphi(a)} Z_{c\alpha}(1, \chi_{o,\frac{a}{c}}) \cdot \frac{Q}{c} + O(Q \cdot e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}}),$$

where we put $c = (a, b)$ and $\chi_{o,\frac{a}{c}}$ is the principal character mod $\frac{a}{c}$.

Since almost all irrational α (including all algebraic irrational α) satisfy the above condition, we get the following.

THEOREM I-3′. *Let $Q > Q_o$. For almost all irrational α (including all algebraic irrational α) in $0 \le \alpha \le 1$, we have for any $1 \le b \le a$*

$$F_\alpha(a, b, Q) - \alpha F(a, b, Q) = -\frac{1}{2a}(Q \log Q - Q) - \frac{1}{\varphi(a)} Z_{c\alpha}(1, \chi_{o,\frac{a}{c}}) \cdot \frac{Q}{c} + O(Q \cdot e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}}),$$

where we put $c = (a, b)$ and $\chi_{o,\frac{a}{c}}$ is the principal character mod $\frac{a}{c}$.

For a rational α, we shall prove the following.

THEOREM I-4. *Let $Q > Q_o$. Suppose that $\alpha = \frac{p}{q} (> 0)$ with $(p, q) = 1$. Then we have for any $1 \le b \le a$*

$$\begin{aligned} F_\alpha \quad &(a, b, Q) - \alpha F(a, b, Q) \\ &= \Big(-\frac{1}{2a} + C_1\Big(a, b, \frac{p}{q}\Big)\Big) Q \log Q + \Big(\frac{1}{2a} + C_2\Big(a, b, \frac{p}{q}\Big)\Big) Q \\ &\quad + O(Qe^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}}), \end{aligned}$$

where we put

$$C_1\Big(a, b, \frac{p}{q}\Big) = -\frac{1}{\varphi\big(\frac{a}{c}\big)\big[\frac{q}{d_1}, \frac{a}{c}\big]c} \sum_{h=1}^{\frac{q}{d_1}} \left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\,(\frac{q}{d_1}, \frac{a}{c}))}}^{\frac{a}{c}} \chi_{o, \frac{a}{c}}(j),$$

$$\begin{aligned} C_2\Big(a, b, \frac{p}{q}\Big) &= C_1\Big(a, b, \frac{p}{q}\Big) \cdot \Big(-\sum_{p|a} \frac{\log p}{p-1} - \gamma_o - \log\Big(\Big[\frac{q}{d_1}, \frac{a}{c}\Big]c\Big) - 1\Big) + C_3\Big(a, b, \frac{p}{q}\Big) \\ &\quad + C_4\Big(a, b, \frac{p}{q}\Big) - \Big(1 - \delta_1(c)\Big)\Big(C_5(a, b, \frac{p}{q}) + C_6\Big(a, b, \frac{p}{q}\Big)\Big) \end{aligned}$$

with

$$C_3\Big(a, b, \frac{p}{q}\Big) = \frac{1}{\varphi\big(\frac{a}{c}\big)\big[\frac{q}{d_1}, \frac{a}{c}\big]c} \sum_{h=1}^{\frac{q}{d_1}} \left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\,(\frac{q}{d_1}, \frac{a}{c}))}}^{\frac{a}{c}} \chi_{o, \frac{a}{c}}(j) \left(\frac{\Gamma'}{\Gamma}\left(\frac{\rho(h, j)}{\big[\frac{q}{d_1}, \frac{a}{c}\big]}\right) + \frac{\big[\frac{q}{d_1}, \frac{a}{c}\big]}{\rho(h, j)}\right),$$

$$\begin{aligned} C_4\Big(a, b, \frac{p}{q}\Big) &= \frac{1}{\varphi\big(\frac{a}{c}\big)\big[\frac{q}{d_1}, \frac{a}{c}\big]c} \sum_{\chi \bmod \frac{a}{c}, \chi \ne \chi_{o, \frac{a}{c}}} \bar{\chi}\Big(\frac{b}{c}\Big) \frac{L'(1, \chi\chi_{o,c})}{L(1, \chi\chi_{o,c})} \\ &\quad \cdot \sum_{h=1}^{\frac{q}{d_1}} \left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\,(\frac{q}{d_1}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j), \end{aligned}$$

$$\begin{aligned} C_5\Big(a, b, \frac{p}{q}\Big) &= \frac{1}{\varphi\big(\frac{a}{c}\big)\big[\frac{q}{d_2}, \frac{a}{c}\big]c} \sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\Big(\frac{b}{c}\Big) \sum_{k=1}^{r} \delta_2(\nu_k) \log p_k \\ &\quad \cdot \sum_{\mu_k=1}^{\nu_k-1} \sum_{h=1}^{\frac{q}{d_2}} \left(\left\{\frac{\frac{c}{p_k^{\mu_k} d_2}p}{q/d_2}h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\,(\frac{q}{d_2}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j) \end{aligned}$$

and

$$C_6\left(a, b, \frac{p}{q}\right) = \frac{1}{\varphi\left(\frac{a}{c}\right)\left[\frac{q}{d_3}, \frac{a}{c}\right]c} \sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{k=1}^{r} \log p_k \frac{p_k}{p_k - \chi(p_k)}$$

$$\cdot \sum_{h=1}^{\frac{q}{d_3}} \left(\left\{ \frac{\frac{c}{p_k^{\nu_k} d_3} p}{q/d_3} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \ (\bmod\, (\frac{q}{d_3}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j)\,,$$

$c = (a, b)$, $d_1 = (c, q)$, $d_2 = \left(\frac{c}{p_k^{\mu_k}}, q\right)$, $d_3 = \left(\frac{c}{p_k^{\nu_k}}, q\right)$, $\chi_{o,a}$ *is the principal character mod* a, $\delta_1(c)$ *is introduced in Lemma 1 above,*

$$\delta_2(\nu_k) = \begin{cases} 1 & \textit{if } \nu_k > 1 \\ 0 & \textit{if } \nu_k = 1\,, \end{cases}$$

we write for $c > 1$

$$c = \prod_{k=1}^{r} p_k^{\nu_k}$$

with prime numbers $p_1, \ldots, p_r$ *and integers* $\nu_1, \ldots, \nu_r (\geq 1)$, $\Gamma(s)$ *is the* Γ*−function and* $\rho(h, j)$ *is the integer determined by the congruence conditions*

$$\begin{cases} x \equiv h & \left(\bmod\ \frac{q}{d}\right) \\ x \equiv j & \left(\bmod\ \frac{a}{c}\right). \end{cases}$$

We mention one example of the constant $C_1\left(a, b, \frac{p}{q}\right)$.

EXAMPLE. For any rational number $0 < \frac{p}{q} \leq 1$ with $(p, q) = 1$ and for $b = 1, 2, 3$, we have

$$C_1\left(3, b, \frac{p}{q}\right) = \begin{cases} 0 & \text{if } 3 \mid q \text{ and } b = 1, 2 \\ \frac{1}{6q} & \text{if } 3 \not| q \text{ and } b = 1, 2 \\ \frac{1}{2q} & \text{if } 3 \mid q \text{ and } b = 3 \\ \frac{1}{6q} & \text{if } 3 \not| q \text{ and } b = 3 \end{cases}$$

When $(a, b) = 1$, the result is simpler. We state it as follows.

COROLLARY 1. *Let* $Q > Q_o$. *Suppose that* $\alpha = \frac{p}{q}$ *with* $(p, q) = 1$. *Then for any* $1 \leq b \leq a$ *with* $(a, b) = 1$, *we have*

$$F_\alpha(a, b, Q) - \alpha F(a, b, Q) = \left(-\frac{1}{2a} + C_1'\left(a, b, \frac{p}{q}\right)\right) Q \log Q + \left(\frac{1}{2a} + C_2'\left(a, b, \frac{p}{q}\right)\right) Q$$

$$+ O(Qe^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}})\,,$$

where we put

$$C_1'\left(a,b,\frac{p}{q}\right)=-\frac{1}{\varphi(a)[q,a]}\sum_{h=1}^{q}\left(\left\{\frac{p}{q}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h\ (mod\,(q,a))}}^{a}\chi_{o,a}(j)$$

and

$$C_2'\left(a,b,\frac{p}{q}\right)=C_1'\left(a,b,\frac{p}{q}\right)\left(-\sum_{p|a}\frac{\log p}{p-1}-\gamma_o-\log[q,a]-1\right)+C_3'\left(a,b,\frac{p}{q}\right)$$
$$+\frac{1}{\varphi(a)[q,a]}\sum_{\chi\ mod\,a,\,\chi\neq\chi_{o,a}}\bar{\chi}(b)\frac{L'(1,\chi)}{L(1,\chi)}\sum_{h=1}^{q}\left(\left\{\frac{p}{q}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h\ (mod\,(q,a))}}^{a}\chi(j)$$

with

$$C_3'\left(a,b,\frac{p}{q}\right)=\frac{1}{\varphi(a)[q,a]}\sum_{h=1}^{q}\left(\left\{\frac{p}{q}h\right\}-\frac{1}{2}\right)$$
$$\times\sum_{\substack{j=1\\ j\equiv h\ (mod\,(q,a))}}^{a}\chi_{o,a}(j)\left(\frac{\Gamma'}{\Gamma}\left(\frac{\rho(h,j)}{[q,a]}\right)+\frac{[q,a]}{\rho(h,j)}\right).$$

Under the Generalized Riemann Hypothesis, we can refine Theorems I-3$'$ and I-4 as follows. We shall omit the proofs of these theorems.

THEOREM I-5. (On G.R.H.). *Let $Q>Q_o$. For almost all irrational α (including all algebraic irrational α) in $0\le\alpha\le1$, we have for any $1\le b\le a$ and for any $\varepsilon>0$*

$$F_\alpha(a,b,Q)-\alpha F(a,b,Q)$$
$$=-\frac{1}{2a}(Q\log Q-Q)-\frac{1}{\varphi(a)}Z_{c\alpha}(1,\chi_{o,\frac{a}{c}})\cdot\frac{Q}{c}+O(Q^{\frac{1}{2}+\varepsilon}),$$

THEOREM I-6. (On G.R.H.). *Let $Q>Q_o$. Suppose that $\alpha=\frac{p}{q}$ with $(p,q)=1$. Then we have for any $1\le b\le a$ and for any $\varepsilon>0$*

$$F_\alpha(a,b,Q)-\alpha F(a,b,Q)$$
$$=\left(-\frac{1}{2a}+C_1\left(a,b,\frac{p}{q}\right)\right)Q\log Q+\left(\frac{1}{2a}+C_2\left(a,b,\frac{p}{q}\right)\right)Q+O(Q^{\frac{1}{2}+\varepsilon}),$$

where $C_1\left(a,b,\frac{p}{q}\right)$ and $C_1\left(a,b,\frac{p}{q}\right)$ are same as in Theorem I-4.

In the section 2, we shall prove a preliminary lemma for the proof of Theorems I-1 and I-2. We shall prove Lemma 1 in the section 3, Theorem I-1 in the section 4 and Theorem I-2 in the section 5. Some of the details of the proof of Theorem I-2 will be omitted. In the section 6, we shall give some preliminaries for the proof of Theorems I-3 and I-4. We shall prove Theorem I-3 in the section 7 and Theorem I-4 in the section 8.

§I-2. Preliminary lemma for the proof of Theorems I-1 and I-2

We notice first the following lemma.

LEMMA 2. *For any $Q \geq 1$, for any α in $0 \leq \alpha \leq 1$ and for any $1 \leq b \leq a$, we have the following.*

$$(\mathrm{i}) \quad F_\alpha(a, b, Q) = \sum_{\substack{n \leq Q, \\ n \equiv b \,(\mathrm{mod}\, a)}} \sum_{h \leq \alpha n} \log(h, n)\,.$$

$$(\mathrm{ii}) \quad F_\alpha(a, b, Q) - \alpha F(a, b, Q) + \frac{1}{2} \sum_{\substack{n \leq Q \\ n \equiv b \,(\mathrm{mod}\, a)}} \log n = - \sum_{\substack{dm \leq Q \\ dm \equiv b \,(\mathrm{mod}\, a)}} \Lambda(d)\left(\{\alpha m\} - \frac{1}{2}\right).$$

(iii) *When $(a, b) = 1$, then we have*

$$F_\alpha(a, b, Q) - \alpha F(a, b, Q) + \frac{1}{2} \sum_{\substack{n \leq Q \\ n \equiv b \,(\mathrm{mod}\, a)}} \log n$$

$$= -\frac{Q}{\varphi(a)} \sum_{\substack{m \leq Q \\ (m,a)=1}} \frac{\{\alpha m\} - \frac{1}{2}}{m} - \sum_{\substack{v=1 \\ (v,a)=1}}^{a} \sum_{\substack{m \leq Q, (m,a)=1 \\ m \equiv \bar{v} \,(\mathrm{mod}\, a)}} \left(\{\alpha m\} - \frac{1}{2}\right) R\left(\frac{Q}{m}, a, bv\right)$$

and

$$F(\alpha; a, b, Q) = - \sum_{\substack{v=1 \\ (v,a)=1}}^{a} \sum_{\substack{m \leq Q, (m,a)=1 \\ m \equiv \bar{v} \,(\mathrm{mod}\, a)}} \left(\{\alpha m\} - \frac{1}{2}\right) R\left(\frac{Q}{m}, a, bv\right),$$

where $\bar{v}$ satisfies $v\bar{v} \equiv 1 (\mathrm{mod}\, a)$ for $(v, a) = 1$ and we put for any $X \geq 1$

$$R(X, a, b) = \sum_{\substack{d \leq X \\ d \equiv b \,(\mathrm{mod}\, a)}} \Lambda(d) - \frac{1}{\varphi(a)} X\,.$$

(Proof of Lemma 2)

We notice first that

$$\sum_{\substack{n \leq Q, \\ n \equiv b \,(\mathrm{mod}\, a)}} \sum_{h \leq \alpha n} \log(h, n) = \sum_{m \leq Q} \log m \sum_{\substack{n \leq Q, h \leq \alpha n \\ n \equiv b \,(\mathrm{mod}\, a), (h,n)=m}} 1$$

$$= \sum_{m \leq Q} \log m \sum_{\substack{mn \leq Q, h \leq \alpha n \\ mn \equiv b \,(\mathrm{mod}\, a), (h,n)=1}} 1 = \sum_{\substack{n \leq Q, h \leq \alpha n \\ (h,n)=1}} \sum_{\substack{m \leq \frac{Q}{n} \\ mn \equiv b \,(\mathrm{mod}\, a)}} \log m$$

$$= \sum_{\substack{i=1, f_i = \frac{a_i}{q_i} \in F_Q \\ f_i \leq \alpha}}^{A} \left(\sum_{\substack{m \leq \frac{Q}{q_i} \\ mq_i \equiv b \,(\mathrm{mod}\, a)}} \log m \right) = F_\alpha(a, b, Q)\,.$$

Thus we have

$$F_\alpha(a,b,Q)=\sum_{\substack{n\le Q,\\ n\equiv b\,(\mathrm{mod}\,a)}}\sum_{h\le\alpha n}\sum_{d|(h,n)}\Lambda(d)=\sum_{d\le Q}\Lambda(d)\sum_{\substack{d|n\le Q\\ n\equiv b\,(\mathrm{mod}\,a)}}\sum_{d|h\le\alpha n}1$$

$$=\sum_{d\le Q}\Lambda(d)\sum_{\substack{m\le\frac{Q}{d}\\ dm\equiv b\,(\mathrm{mod}\,a)}}\sum_{h\le\alpha m}1=\sum_{d\le Q}\Lambda(d)\sum_{\substack{m\le\frac{Q}{d}\\ dm\equiv b\,(\mathrm{mod}\,a)}}[\alpha m]$$

$$=\alpha\sum_{d\le Q}\Lambda(d)\sum_{\substack{m\le\frac{Q}{d}\\ dm\equiv b\,(\mathrm{mod}\,a)}}m-\sum_{d\le Q}\Lambda(d)\sum_{\substack{m\le\frac{Q}{d}\\ dm\equiv b\,(\mathrm{mod}\,a)}}\left(\{\alpha m\}-\frac{1}{2}\right)$$

$$-\frac{1}{2}\sum_{d\le Q}\Lambda(d)\sum_{\substack{m\le\frac{Q}{d}\\ dm\equiv b\,(\mathrm{mod}\,a)}}1$$

$$=\alpha F(a,b,Q)-\sum_{d\le Q}\Lambda(d)\sum_{\substack{m\le\frac{Q}{d}\\ dm\equiv b\,(\mathrm{mod}\,a)}}\left(\{\alpha m\}-\frac{1}{2}\right)-\frac{1}{2}\sum_{\substack{n\le Q\\ n\equiv b\,(\mathrm{mod}\,a)}}\log n\,.$$

We now suppose that $(a,b)=1$ and prove (iii).

$$F_\alpha(a,b,Q)-\alpha F(a,b,Q)+\frac{1}{2}\sum_{\substack{n\le Q\\ n\equiv b\,(\mathrm{mod}\,a)}}\log n=-\sum_{\substack{dm\le Q\\ dm\equiv b\,(\mathrm{mod}\,a)}}\Lambda(d)\left(\{\alpha m\}-\frac{1}{2}\right)$$

$$=-\sum_{m\le Q}\left(\{\alpha m\}-\frac{1}{2}\right)\sum_{\substack{d\le\frac{Q}{m}\\ dm\equiv b\,(\mathrm{mod}\,a)}}\Lambda(d)$$

$$=-\sum_{\substack{v=1\\ (v,a)=1}}^{a}\sum_{\substack{m\le Q,(m,a)=1\\ m\equiv\bar v\,(\mathrm{mod}\,a)}}\left(\{\alpha m\}-\frac{1}{2}\right)\sum_{\substack{d\le\frac{Q}{m}\\ d\equiv bv\,(\mathrm{mod}\,a)}}\Lambda(d)$$

$$=-\sum_{\substack{v=1\\ (v,a)=1}}^{a}\sum_{\substack{m\le Q,(m,a)=1\\ m\equiv\bar v\,(\mathrm{mod}\,a)}}\left(\{\alpha m\}-\frac{1}{2}\right)\left(\frac{1}{\varphi(a)}\frac{Q}{m}+R\left(\frac{Q}{m},a,bv\right)\right)$$

$$=-\frac{Q}{\varphi(a)}\sum_{\substack{v=1\\ (v,a)=1}}^{a}\sum_{\substack{m\le Q,(m,a)=1\\ m\equiv\bar v\,(\mathrm{mod}\,a)}}\frac{\{\alpha m\}-\frac{1}{2}}{m}$$

$$-\sum_{\substack{v=1\\ (v,a)=1}}^{a}\sum_{\substack{m\le Q,(m,a)=1\\ m\equiv\bar v\,(\mathrm{mod}\,a)}}\left(\{\alpha m\}-\frac{1}{2}\right)R\left(\frac{Q}{m},a,bv\right)$$

$$= -\frac{Q}{\varphi(a)} \sum_{\substack{m\le Q\\(m,a)=1}} \frac{\{\alpha m\}-\frac{1}{2}}{m} - \sum_{\substack{v=1\\(v,a)=1}}^{a} \sum_{\substack{m\le Q,(m,a)=1\\ m\equiv\bar{v}\,(\mathrm{mod}\,a)}} \left(\{\alpha m\}-\frac{1}{2}\right) R\left(\frac{Q}{m}, a, bv\right).$$

Consequently, we get

$$F_\alpha(a,b,Q) \quad -\alpha F(a,b,Q) + \frac{1}{2} \sum_{\substack{n\le Q\\ n\equiv b\,(\mathrm{mod}\,a)}} \log n$$

$$= -\frac{Q}{\varphi(a)} \sum_{\substack{m\le Q\\(m,a)=1}} \frac{\{\alpha m\}-\frac{1}{2}}{m} - \sum_{\substack{v=1\\(v,a)=1}}^{a} \sum_{\substack{m\le Q,(m,a)=1\\ m\equiv\bar{v}\,(\mathrm{mod}\,a)}} \left(\{\alpha m\}-\frac{1}{2}\right) R\left(\frac{Q}{m}, a, bv\right).$$

This proves (iii).

§I-3. Proof of Lemma 1

We put $c=(a,b)$ as in the satement of Lemma 1. For the proof of Lemma 1, we see that

$$\begin{aligned}
F(a,b,Q) &= \sum_{d\le Q} \Lambda(d) \sum_{m\le \frac{Q}{d}, md\equiv b \ (\mathrm{mod}\ a)} m \\
&= \sum_{\delta|c} \sum_{d\le Q,(a,d)=\delta} \Lambda(d) \sum_{m\le\frac{Q}{d}, m\frac{d}{\delta}\equiv\frac{b}{\delta}\ (\mathrm{mod}\ \frac{a}{\delta})} m \\
&= \sum_{\delta|c} \sum_{d\le Q,(a,d)=\delta} \Lambda(d) \sum_{j\frac{c}{\delta}\le\frac{Q}{d}, j\frac{d}{\delta}\equiv\frac{b}{c}\ (\mathrm{mod}\ \frac{a}{c})} j\cdot\frac{c}{\delta} \\
&= \sum_{\delta|c} \frac{c}{\delta} \sum_{d\le Q,(a,d)=\delta} \Lambda(d) \sum_{j\le\frac{Q}{d\cdot\frac{c}{\delta}}, j\frac{d}{\delta}\equiv\frac{b}{c}\ (\mathrm{mod}\ \frac{a}{c})} j \\
&= \frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\chi \ \mathrm{mod}\ \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{\delta|c}\frac{c}{\delta} \sum_{d\le Q,(a,d)=\delta} \Lambda(d)\chi\left(\frac{d}{\delta}\right) \sum_{j\le\frac{Q}{d\cdot\frac{c}{\delta}}} j\chi(j) \\
&= \frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\delta|c}\frac{c}{\delta} \sum_{d\le Q,(a,d)=\delta} \Lambda(d) \sum_{j\le\frac{Q}{d\cdot\frac{c}{\delta}}, (j,\frac{a}{c})=1} j \\
&\quad + \frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\chi \ \mathrm{mod}\ \frac{a}{c}, \chi\ne\chi_{o,\frac{a}{c}}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{\delta|c}\frac{c}{\delta} \sum_{d\le Q,(a,d)=\delta} \Lambda(d)\chi\left(\frac{d}{\delta}\right) \sum_{j\le\frac{Q}{d\cdot\frac{c}{\delta}}} j\chi(j) \\
&= U_1 + U_2\,, \quad \text{say}\,,
\end{aligned}$$

where χ runs over all Dirichlet characters mod $\frac{a}{c}$ and $\chi_{o,\frac{a}{c}}$ is the principal character mod $\frac{a}{c}$.

We shall evaluate U_1 first.

$$
\begin{aligned}
U_1 &= \frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\delta|c}\frac{c}{\delta}\sum_{d\le Q,(a,d)=\delta}\Lambda(d)\sum_{j\le\frac{Q}{d\cdot\frac{c}{\delta}}} j\sum_{v|(j,\frac{a}{c})}\mu(v)\\
&= \frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\delta|c}\frac{c}{\delta}\sum_{d\le Q,(a,d)=\delta}\Lambda(d)\sum_{v|\frac{a}{c}}\mu(v)\sum_{v|j\le\frac{Q}{d\cdot\frac{c}{\delta}}} j\\
&= \frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\delta|c}\frac{c}{\delta}\sum_{d\le Q,(a,d)=\delta}\Lambda(d)\sum_{v|\frac{a}{c}}\mu(v)v\sum_{k\le\frac{Q}{d\cdot\frac{c}{\delta}v}} k\\
&= \frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\delta|c}\frac{c}{\delta}\sum_{d\le Q,(a,d)=\delta}\Lambda(d)\sum_{v|\frac{a}{c}}\mu(v)v\frac{1}{2}\left[\frac{Q}{d\cdot\frac{c}{\delta}v}\right]\left(\left[\frac{Q}{d\cdot\frac{c}{\delta}v}\right]+1\right)\\
&= \frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\delta|c}\frac{c}{\delta}\sum_{d\le Q,(a,d)=\delta}\Lambda(d)\sum_{v|\frac{a}{c}}\mu(v)v\left(\frac{1}{2}\left(\frac{Q}{d\cdot\frac{c}{\delta}v}\right)^2+O\left(\frac{Q}{d\cdot\frac{c}{\delta}v}\right)\right)\\
&= \frac{1}{2}\frac{Q^2}{\varphi(\frac{a}{c})}\sum_{\delta|c}\frac{1}{\frac{c}{\delta}}\sum_{d\le Q,(a,d)=\delta}\frac{\Lambda(d)}{d^2}\sum_{v|\frac{a}{c}}\frac{\mu(v)}{v}\\
&\quad +O\Big(\frac{Q}{\varphi\left(\frac{a}{c}\right)}\sum_{\delta|c}\sum_{d\le Q,(a,d)=\delta}\frac{|\ \Lambda(d)\ |}{d}\sum_{v|\frac{a}{c}}|\ \mu(v)\ |\Big) = U_3 + O(U_4)\,, \quad \text{say}.
\end{aligned}
$$

We notice first that

$$
U_4 \ll \frac{Q}{\varphi\left(\frac{a}{c}\right)}\tau\left(\frac{a}{c}\right)\sum_{\delta|c}\sum_{\delta|d\le Q}\frac{\Lambda(d)}{d} \ll \frac{Q}{\varphi\left(\frac{a}{c}\right)}\tau\left(\frac{a}{c}\right)\left(\sum_{j\le Q}\frac{\Lambda(j)}{j}+\sum_{\substack{\delta|c\\ \delta>1}}\sum_{j\le\frac{Q}{\delta}}\frac{\Lambda(\delta j)}{\delta j}\right),
$$

where we put $\tau(n)=\sum_{d|n}1$. We notice that

$$
\sum_{j\le Q}\frac{\Lambda(j)}{j} \ll \log Q\,.
$$

When $c=p_1^{\nu_1}\cdots p_k^{\nu_k}\cdots p_r^{\nu_r}$ with prime numbers p_k and integers $\nu_k\ge 1$, we have

$$
\begin{aligned}
\sum_{\substack{\delta|c\\ \delta>1}}\sum_{j\le\frac{Q}{\delta}}\frac{\Lambda(\delta j)}{\delta j} &\ll \sum_{k=1}^{r}\log p_k\sum_{\mu_k=1}^{\nu_k}\frac{1}{p_k^{\mu_k}}\sum_{\substack{p_k^{\eta_k}\le\frac{Q}{p_k^{\mu_k}}\\ \eta_k\ge 0}}\frac{1}{p_k^{\eta_k}}\\
&\ll \sum_{k=1}^{r}\log p_k\sum_{\mu_k=1}^{\nu_k}\frac{1}{p_k^{\mu_k}}\frac{1}{1-\frac{1}{p_k}} \ll \log c\,.
\end{aligned}
$$

Hence, we get

$$
U_4 \ll \frac{Q}{\varphi\left(\frac{a}{c}\right)}\tau\left(\frac{a}{c}\right)(\log Q+\log c)\,.
$$

We notice next that

$$\begin{aligned}U_3 &= \frac{1}{2}\frac{Q^2}{a}\sum_{\delta|c}\delta\sum_{d\le Q,(a,d)=\delta}\frac{\Lambda(d)}{d^2}\\ &= \frac{1}{2}\frac{Q^2}{a}\sum_{\delta|c}\delta\sum_{d=1,(a,d)=\delta}^{\infty}\frac{\Lambda(d)}{d^2} + O\bigg(\frac{Q^2}{a}\sum_{\delta|c}\delta\sum_{d>Q,(a,d)=\delta}\frac{|\Lambda(d)|}{d^2}\bigg)\\ &= U_5 + O(U_6)\,,\quad \text{say}\,,\end{aligned}$$

where

$$\begin{aligned}U_6 &\ll \frac{Q^2}{a}\sum_{\delta|c}\frac{1}{\delta}\sum_{j>\frac{Q}{\delta}}\frac{\Lambda(\delta j)}{j^2} \ll \frac{Q^2}{a}\bigg(\sum_{j>Q}\frac{\Lambda(j)}{j^2} + (1-\delta_1(c))\sum_{\substack{\delta|c\\ \delta>1}}\frac{1}{\delta}\sum_{j>\frac{Q}{\delta}}\frac{\Lambda(\delta j)}{j^2}\bigg)\\ &\ll \frac{Q}{a}\bigg(1+(1-\delta_1(c))\frac{1}{Q}\sum_{k=1}^{r}p_k^{\nu_k}\log p_k\bigg)\,,\end{aligned}$$

where $\delta_1(c)$ is introduced in the statement of Lemma 1. We get also

$$\begin{aligned}U_5 &= \frac{1}{2}\frac{Q^2}{a}\sum_{d=1,(a,d)=1}^{\infty}\frac{\Lambda(d)}{d^2} + \frac{1}{2}\frac{Q^2}{a}(1-\delta_1(c))\sum_{\substack{\delta|c\\ \delta>1}}\delta\sum_{d=1,(a,d)=\delta}^{\infty}\frac{\Lambda(d)}{d^2}\\ &= U_7 + U_8\,,\quad \text{say}\,.\end{aligned}$$

$$U_7 = -\frac{Q^2}{2a}\frac{L'(2,\chi_{o,a})}{L(2,\chi_{o,a})} = -\frac{3Q^2}{\pi^2 a\prod_{p|a}\left(1-\frac{1}{p^2}\right)}L'(2,\chi_{o,a})\,,$$

where $\chi_{o,a}$ is the principal character mod a.

$$\begin{aligned}U_8 &= \frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}\sum_{\mu_k=1}^{\nu_k}p_k^{\mu_k}\sum_{d=1,(a,d)=p_k^{\mu_k}}^{\infty}\frac{\Lambda(d)}{d^2}\\ &= \frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}\log p_k\sum_{\mu_k=1}^{\nu_k}p_k^{\mu_k}\sum_{\substack{\eta_k=0\\ (\frac{a}{p_k^{\mu_k}},p_k^{\eta_k})=1}}^{\infty}\frac{1}{p_k^{2\mu_k+2\eta_k}}\\ &= \frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}\log p_k\sum_{\mu_k=1}^{\nu_k}\frac{1}{p_k^{\mu_k}}\sum_{\eta_k=0}^{\infty}\left(\frac{\chi_{o,\frac{a}{p_k^{\mu_k}}}(p_k)}{p_k^2}\right)^{\eta_k}\\ &= \frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}\log p_k\sum_{\mu_k=1}^{\nu_k}\frac{1}{p_k^{\mu_k}}\frac{p_k^2}{p_k^2-\chi_{o,\frac{a}{p_k^{\mu_k}}}(p_k)}\\ &= \frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}(1-\delta_2(\nu_k))\cdot\log p_k\frac{1}{p_k^{\nu_k}}\frac{p_k^2}{p_k^2-\chi_{o,\frac{a}{p_k^{\nu_k}}}(p_k)}\end{aligned}$$

$$
\begin{aligned}
&+\frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}\delta_2(\nu_k)\cdot\log p_k\sum_{\mu_k=1}^{\nu_k}\frac{1}{p_k^{\mu_k}}\frac{p_k^2}{p_k^2-\chi_{o,\frac{a}{p_k^{\mu_k}}}(p_k)}\\
&=\frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}(1-\delta_2(\nu_k))\cdot\log p_k\frac{1}{p_k^{\nu_k}}\frac{p_k^2}{p_k^2-\chi_{o,\frac{a}{p_k^{\nu_k}}}(p_k)}\\
&+\frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}\delta_2(\nu_k)\cdot\log p_k\sum_{\mu_k=1}^{\nu_k-1}\frac{1}{p_k^{\mu_k}}\\
&+\frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}\delta_2(\nu_k)\cdot\log p_k\frac{1}{p_k^{\nu_k}}\frac{p_k^2}{p_k^2-\chi_{o,\frac{a}{p_k^{\nu_k}}}(p_k)}\\
&=\frac{Q^2}{2a}(1-\delta_1(c))\sum_{k=1}^{r}\frac{\log p_k}{p_k^{\nu_k}}\left(\frac{p_k^{\nu_k}-p_k}{p_k-1}+\frac{p_k^2}{p_k^2-\chi_{o,\frac{a}{p_k^{\nu_k}}}(p_k)}\right),
\end{aligned}
$$

where $\delta_2(\nu_k)$ is introduced in the statement of Theorem I-4.

Consequently, we get

$$
\begin{aligned}
U_1=\frac{Q^2}{2a}\Bigg(&-\frac{6}{\pi^2a\prod_{p|a}\left(1-\frac{1}{p^2}\right)}L'(2,\chi_{o,a})\\
&+(1-\delta_1(c))\sum_{k=1}^{r}\frac{\log p_k}{p_k^{\nu_k}}\left(\frac{p_k-p_k^{\nu_k}}{1-p_k}+\frac{p_k^2}{p_k^2-\chi_{o,\frac{a}{p_k^{\nu_k}}}(p_k)}\right)\Bigg)\\
&+O\left(\frac{Q}{\varphi\left(\frac{a}{c}\right)}\tau\left(\frac{a}{c}\right)\left(\log Q+\log c\right)\right)\\
&+O\left(\frac{Q}{a}\left(1+(1-\delta_1(c))\frac{1}{Q}\sum_{k=1}^{r}p_k^{\nu_k}\log p_k\right)\right).
\end{aligned}
$$

Finally, we shall estimate U_2. To treat U_2, we may suppose that $\frac{a}{c}\geq 3$. By the partial summation, we get

$$
\sum_{j\leq\frac{Q}{d\cdot\frac{c}{\delta}}} j\chi(j)\ll\frac{Q}{d\cdot\frac{c}{\delta}}\sqrt{\frac{a}{c}}\log\left(\frac{a}{c}+2\right),
$$

where we have used Polya-Vinogradov's theorem (cf. Theorem 9.18 on p. 307 of Montgomery-Vaughan [30]) which states that for any non-principal character mod $\frac{a}{c}$ and for $y\geq 1$ we have

$$
\sum_{n\leq y}\chi(n)\ll\sqrt{\frac{a}{c}}\log\left(\frac{a}{c}+2\right).
$$

Hence, we get as in the estimate of U_4

$$\begin{aligned} U_2 &\ll \frac{Q}{\varphi\left(\frac{a}{c}\right)} \sum_{\chi \bmod \frac{a}{c}, \chi \neq \chi_{o,\frac{a}{c}}} \sum_{\delta | c} \sum_{d \le Q, (a,d)=\delta} \frac{\Lambda(d)}{d} \sqrt{\frac{a}{c}} \log\left(\frac{a}{c}+2\right) \\ &\ll Q\sqrt{\frac{a}{c}} \log\left(\frac{a}{c}+2\right)(\log Q + \log c)\,. \end{aligned}$$

Combining all of these estimates, we get

$$\begin{aligned} F(a,b,Q) = \frac{Q^2}{2a}\Bigg(&-\frac{6}{\pi^2 a \prod_{p|a}\left(1-\frac{1}{p^2}\right)} L'(2,\chi_{o,a}) \\ &+(1-\delta_1(c)) \sum_{k=1}^{r} \frac{\log p_k}{p_k^{\nu_k}} \left(\frac{p_k - p_k^{\nu_k}}{1-p_k} + \frac{p_k^2}{p_k^2 - \chi_{o,\frac{a}{p_k^{\nu_k}}}(p_k)} \right)\Bigg) \\ &+O(Q\sqrt{\frac{a}{c}} \log\left(\frac{a}{c}+2\right)(\log Q + \log c)) \\ &+O\left(\frac{Q}{a}\left(1 + (1-\delta_1(c)) \frac{1}{Q} \sum_{k=1}^{r} p_k^{\nu_k} \log p_k \right)\right). \end{aligned}$$

This proves our Lemma 1.

§I-4. Proof of Theorem I-1

We consider the following integral, using (iii) of Lemma 2 first,

$$\begin{aligned} &\int_0^1 F(\alpha; a, b, Q) e^{2\pi i \alpha}\, d\alpha \\ &= -\sum_{\substack{v=1\\(v,a)=1}}^{a} \sum_{\substack{n \le Q, (n,a)=1\\ n \equiv \bar{v} \,(\mathrm{mod}\, a)}} R\left(\frac{Q}{n}, a, bv\right) \int_0^1 \left(\{\alpha n\} - \frac{1}{2}\right) e^{2\pi i \alpha}\, d\alpha \\ &= -\sum_{\substack{v=1\\(v,a)=1}}^{a} \sum_{\substack{n \le Q, (n,a)=1\\ n \equiv \bar{v} \,(\mathrm{mod}\, a)}} \frac{1}{n} R\left(\frac{Q}{n}, a, bv\right) \int_0^n \left(\{y\} - \frac{1}{2}\right) e^{2\pi i \frac{y}{n}}\, dy \\ &= -\sum_{\substack{v=1\\(v,a)=1}}^{a} \sum_{\substack{n \le Q, (n,a)=1\\ n \equiv \bar{v} \,(\mathrm{mod}\, a)}} \frac{1}{n} R\left(\frac{Q}{n}, a, bv\right) \sum_{j=0}^{n-1} \int_j^{j+1} \left(\{y\} - \frac{1}{2}\right) e^{2\pi i \frac{y}{n}}\, dy \\ &= -\sum_{\substack{v=1\\(v,a)=1}}^{a} \sum_{\substack{n \le Q, (n,a)=1\\ n \equiv \bar{v} \,(\mathrm{mod}\, a)}} \frac{1}{n} R\left(\frac{Q}{n}, a, bv\right) \sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n}} \int_0^1 \left(t - \frac{1}{2}\right) e^{2\pi i \frac{t}{n}}\, dt. \end{aligned}$$

Since the last integral is

$$= \frac{n}{4\pi i}\left(e^{2\pi i \frac{1}{n}} + 1\right) - \left(\frac{n}{2\pi i}\right)^2 (e^{2\pi i \frac{1}{n}} - 1)\,,$$

we get

$$\begin{aligned}
&\int_0^1 F(\alpha; a, b, Q) e^{2\pi i \alpha}\, d\alpha \\
&= -\sum_{\substack{v=1 \\ (v,a)=1}}^{a} \sum_{\substack{n \le Q, (n,a)=1 \\ n \equiv \bar{v} \,(\mathrm{mod}\, a)}} \frac{1}{n} R\left(\frac{Q}{n}, a, bv\right) \sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n}} \\
&\qquad \cdot \left(\frac{n}{4\pi i}\left(e^{2\pi i \frac{1}{n}} + 1\right) - \left(\frac{n}{2\pi i}\right)^2 (e^{2\pi i \frac{1}{n}} - 1)\right) \\
&= -\frac{1}{2\pi i} \sum_{\substack{v=1,\, (v,a)=1 \\ 1 \equiv \bar{v} \,(\mathrm{mod}\, a)}}^{a} R(Q, a, bv) = -\frac{1}{2\pi i} R(Q, a, b)\,.
\end{aligned}$$

Hence, we get

$$|R(Q, a, b)|^2 \ll \int_0^1 |F(\alpha; a, b)|^2 \, d\alpha\,.$$

Thus if we assume

$$\int_0^1 |F(\alpha; a, b, Q)|^2 \, d\alpha \ll Q^{1+\varepsilon}\,,$$

for all b satisfying $1 \le b \le a$ and $(b, a) = 1$, then we get

$$R(Q, a, b) \ll Q^{\frac{1}{2}+\varepsilon}$$

for all b satisfying $1 \le b \le a$ and $(b, a) = 1$. This implies G.R.H. for all $L(s, \chi)$ with Dirichlet characters χ mod a.

On the other hand, using (iii) of Lemma 2 again,

$$\begin{aligned}
&\int_0^1 |F(\alpha; a, b, Q)|^2 \, d\alpha \\
&= \sum_{\substack{v=1 \\ (v,a)=1}}^{a} \sum_{\substack{v'=1 \\ (v',a)=1}}^{a} \sum_{\substack{n \le Q, (n,a)=1 \\ n \equiv \bar{v} \,(\mathrm{mod}\, a)}} \sum_{\substack{m \le Q, (m,a)=1 \\ m \equiv \bar{v'} \,(\mathrm{mod}\, a)}} R\left(\frac{Q}{n}, a, bv\right) R\left(\frac{Q}{m}, a, bv'\right) \\
&\qquad \cdot \int_0^1 \left(\{\alpha n\} - \frac{1}{2}\right)\left(\{\alpha m\} - \frac{1}{2}\right) \\
&= \frac{1}{12} \sum_{\substack{v=1 \\ (v,a)=1}}^{a} \sum_{\substack{v'=1 \\ (v',a)=1}}^{a} \sum_{\substack{n \le Q, (n,a)=1 \\ n \equiv \bar{v} \,(\mathrm{mod}\, a)}} \sum_{\substack{m \le Q, (m,a)=1 \\ m \equiv \bar{v'} \,(\mathrm{mod}\, a)}} R\left(\frac{Q}{n}, a, bv\right) R\left(\frac{Q}{m}, a, bv'\right) \frac{(m, n)}{[m, n]}\,.
\end{aligned}$$

If we assume G.R.H., then the right hand side is

$$\ll Q^{1+\varepsilon}\,.$$

This proves Theorem I-1 .

§I-5. Proof of Theorem I-2

We put for $1 \le i \le A$

$$I_i = \left[\frac{a_i + a_{i-1}}{q_i + q_{i-1}}, \frac{a_i + a_{i+1}}{q_i + q_{i+1}}\right],$$

where we suppose that $\frac{a_0}{q_0} = \frac{0}{1}$ and $\frac{a_{A+1}}{q_{A+1}} = \frac{Q+1}{Q}$. We remark the following lemma.

LEMMA 3. *Let $1 \le i \le A$. If $\alpha \in I_i$, then we have*

$$F(\alpha; a, b, Q) - F(f_i; a, b, Q) \ll \frac{Q}{q_i} \log\left(\frac{Q}{q_i} + 2\right),$$

where the implicit constant may depend on a.

Proof. By the definition of $F(\alpha; a, b, Q)$, we notice first that

$$\begin{aligned} &F(\alpha; a, b, Q) - F(f_i; a, b, Q) \\ &= \sum_{\substack{k=1, f_k = \frac{a_k}{q_k} \in F_Q \\ f_k \le \alpha}}^{A} \left(\sum_{\substack{m \le \frac{Q}{q_k} \\ mq_k \equiv b \,(\mathrm{mod}\, a)}} \log m \right) - \sum_{\substack{k=1, f_k = \frac{a_k}{q_k} \in F_Q \\ f_k \le f_i}}^{A} \left(\sum_{\substack{m \le \frac{Q}{q_k} \\ mq_k \equiv b \,(\mathrm{mod}\, a)}} \log m \right) \\ &\quad -(\alpha - f_i) F(a, b, Q) + \frac{Q}{\varphi(a)} \sum_{\substack{m \le Q \\ (m,a)=1}} \frac{\{\alpha m\} - \{f_i m\}}{m}. \end{aligned}$$

Suppose first that $\alpha \in I_i$ and $\alpha > f_i$. We put $\delta = \alpha - f_i$. Then

$$0 < \delta \le \frac{a_i + a_{i+1}}{q_i + q_{i+1}} - \frac{a_i}{q_i} = \frac{1}{q_i(q_i + q_{i+1})} < \frac{1}{q_i Q}$$

and for $n \le Q$,

$$0 \le \{\delta n\} < \frac{n}{q_i Q} \le \frac{1}{q_i}.$$

If $\{f_i n\} + \{\delta n\} \ge 1$, then we have

$$\frac{1}{q_i} \ge \frac{n}{q_i Q} > 1 - \{f_i n\} \ge \frac{1}{q_i}.$$

This gives a contradiction. Hence we have $\{f_i n\} + \{\delta n\} < 1$ and consequently

$$\{\alpha n\} - \{f_i n\} = \{f_i n + \delta n\} - \{f_i n\} = \{\delta n\}.$$

Thus we have

$$0 \le \{\alpha n\} - \{f_i n\} < \frac{n}{q_i Q}.$$

Hence, if $\alpha \in I_i$ and $\alpha > f_i$, then

$$| \ F(\alpha; a, b, Q) \ - \ F(f_i; a, b, Q) \ |$$

$$= \left| -(\alpha - f_i)F(a,b,Q) + \frac{Q}{\varphi(a)} \sum_{\substack{m \le Q \\ (m,a)=1}} \frac{\{\alpha m\} - \{f_i m\}}{m} \right|$$

$$\ll \frac{1}{q_i Q} F(a,b,Q) + \frac{Q}{\varphi(a)} \sum_{\substack{m \le Q \\ (m,a)=1}} \frac{m}{m q_i Q} \ll \frac{Q}{q_i} \log\left(\frac{Q}{q_i} + 2\right).$$

Suppose next that $\alpha \in I_i$, $\alpha < f_i$ and $q_i \not| n$. We put $\delta' = f_i - \alpha$. Then

$$0 \le \delta' \le \frac{a_i}{q_i} - \frac{a_i + a_{i-1}}{q_i + q_{i-1}} = \frac{1}{q_i(q_i + q_{i-1})} < \frac{1}{q_i Q}$$

and for $n \le Q$,

$$0 \le \{\delta' n\} < \frac{n}{q_i Q} \le \frac{1}{q_i}.$$

On the other hand since $q_i \not| n$, we have

$$\{f_i n\} \ge \frac{1}{q_i} \ge \frac{n}{q_i Q} > \{\delta' n\}.$$

Hence we get

$$\{\alpha n\} - \{f_i n\} = \{f_i n - \delta' n\} - \{f_i n\} = -\{\delta' n\}.$$

Thus we get

$$|\{\alpha n\} - \{f_i n\}| = |-\{\delta' n\}| < \frac{n}{q_i Q}.$$

Hence, if $\alpha \in I_i$ and $\alpha < f_i$, then we have

$$|F(\alpha; a,b,Q) - F(f_i; a,b,Q)|$$

$$= \Bigg| \sum_{\substack{m \le \frac{Q}{q_i} \\ m q_i \equiv b \,(\mathrm{mod}\, a)}} \log m - (\alpha - f_i)F(a,b,Q)$$

$$+ \frac{Q}{\varphi(a)} \sum_{\substack{m \le Q \\ (m,a)=1, q_i \not| m}} \frac{(\{\alpha m\} - \{f_i m\})}{m} + \frac{Q}{\varphi(a)} \sum_{\substack{m \le Q \\ (m,a)=1, q_i | m}} \frac{\{\alpha m\} - \{f_i m\}}{m} \Bigg|$$

$$\ll \frac{Q}{q_i} \log\left(\frac{Q}{q_i} + 2\right) + \frac{Q}{q_i} \sum_{j \le \frac{Q}{q_i}} \frac{1}{j} \ll \frac{Q}{q_i} \log\left(\frac{Q}{q_i} + 2\right).$$

If $\alpha = f_i$, then obviously we have

$$F(\alpha; a,b,Q) - F(f_i; a,b,Q) = 0.$$

This proves our lemma.

Using Lemma 3, we can obtain Theorem I-2 in the same manner as in pp. 228–232 of Fujii [18].

§I-6. Preliminaries for the proof of Theorems I-3 and I-4

We put $c = (a, b)$ and

$$G(\alpha) = F_\alpha(a, b, Q) - \alpha F(a, b, Q) + \frac{1}{2} \sum_{\substack{n \le Q \\ n \equiv b \,(\mathrm{mod}\, a)}} \log n\,.$$

By (ii) of Lemma 2 in the section 2,

$$G(\alpha) = -\sum_{\substack{dm \le Q \\ dm \equiv b \,(\mathrm{mod}\, a)}} \Lambda(d)\left(\{\alpha m\} - \frac{1}{2}\right).$$

We rewrite this as follows.

$$\begin{aligned}
G(\alpha) &= -\sum_{\substack{c | dm \le Q \\ \frac{dm}{c} \equiv \frac{b}{c} \ (\mathrm{mod}\ \frac{a}{c})}} \Lambda(d)\left(\{\alpha m\} - \frac{1}{2}\right) \\
&= -\sum_{\delta | c} \sum_{\substack{dm \le \frac{Q}{c} \\ (d, \frac{c}{\delta}) = 1,\, dm \equiv \frac{b}{c} \ (\mathrm{mod}\ \frac{a}{c})}} \Lambda(\delta d)\left(\left\{\alpha \frac{c}{\delta} m\right\} - \frac{1}{2}\right) \\
&= -\frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\delta | c} \sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{dm \le \frac{Q}{c}} \Lambda(\delta d) \chi_{o, \frac{c}{\delta}}(d) \chi(d) \left(\left\{\alpha \frac{c}{\delta} m\right\} - \frac{1}{2}\right) \chi(m) \\
&= -\frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{dm \le \frac{Q}{c}} \Lambda(d) \chi_{o,c}(d) \chi(d) \left(\{\alpha c m\} - \frac{1}{2}\right) \chi(m) \\
&\quad -(1 - \delta_1(c)) \frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\delta | c, \delta > 1} \sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right) \\
&\quad\quad \cdot \sum_{dm \le \frac{Q}{c}} \Lambda(\delta d) \chi_{o, \frac{c}{\delta}}(d) \chi(d) \left(\left\{\alpha \frac{c}{\delta} m\right\} - \frac{1}{2}\right) \chi(m) \\
&= S_1 + S_2\,, \quad \text{say}\,,
\end{aligned}$$

where $\delta_1(c)$ is introduced in the statement of Lemma 1. We rewrite S_2 further as follows. When $c = p_1^{\nu_1} \cdots p_k^{\nu_k} \cdots p_r^{\nu_r}$ with prime numbers p_k and integers $\nu_k \ge 1$, we have

$$\begin{aligned}
S_2 = &-(1 - \delta_1(c)) \frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{k=1}^{r} \sum_{\mu_k = 1}^{\nu_k} \\
&\cdot \sum_{dm \le \frac{Q}{c}} \Lambda(p_k^{\mu_k} d) \chi_{o, \frac{c}{p_k^{\mu_k}}}(d) \chi(d) \left(\left\{\alpha \frac{c}{p_k^{\mu_k}} m\right\} - \frac{1}{2}\right) \chi(m)
\end{aligned}$$

$$
\begin{aligned}
&= -(1-\delta_1(c))\frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\chi \bmod \frac{a}{c}}\bar{\chi}\left(\frac{b}{c}\right)\sum_{k=1}^{r}\sum_{\mu_k=1}^{\nu_k}\\
&\quad\cdot\sum_{\substack{p_k^{\eta_k} m\le \frac{Q}{c}\\ \eta_k\ge 0}} \Lambda(p_k^{\mu_k}p_k^{\eta_k})\chi_{o,\frac{c}{p_k^{\mu_k}}}(p_k^{\eta_k})\chi(p_k^{\eta_k})\left(\left\{\alpha\frac{c}{p_k^{\mu_k}}m\right\}-\frac{1}{2}\right)\chi(m)\\
&= -(1-\delta_1(c))\frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\chi \bmod \frac{a}{c}}\bar{\chi}\left(\frac{b}{c}\right)\sum_{k=1}^{r}\delta_2(\nu_k)\log p_k\\
&\quad\cdot\sum_{\mu_k=1}^{\nu_k-1}\sum_{m\le\frac{Q}{c}}\left(\left\{\alpha\frac{c}{p_k^{\mu_k}}m\right\}-\frac{1}{2}\right)\chi(m)\\
&\quad -(1-\delta_1(c))\frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\chi \bmod \frac{a}{c}}\bar{\chi}\left(\frac{b}{c}\right)\sum_{k=1}^{r}\log p_k\\
&\quad\cdot\sum_{\substack{p_k^{\eta_k} m\le \frac{Q}{c}\\ \eta_k\ge 0}}\chi(p_k^{\eta_k})\left(\left\{\alpha\frac{c}{p_k^{\nu_k}}m\right\}-\frac{1}{2}\right)\chi(m)\\
&= S_3+S_4\,,\quad \text{say}\,,
\end{aligned}
$$

where we put

$$
\delta_2(\nu_k)=\begin{cases}1 & \text{if } \nu_k>1\\ 0 & \text{if } \nu_k=1\,.\end{cases}
$$

We shall evaluate for each χ mod $\frac{a}{c}$ the following sums in the following sections.

$$
M_1(\chi)=\sum_{dm\le\frac{Q}{c}}\Lambda(d)\chi_{o,c}(d)\chi(d)\left(\{\alpha cm\}-\frac{1}{2}\right)\chi(m)\,,
$$

$$
M_2(\chi)=\sum_{m\le\frac{Q}{c}}\left(\left\{\alpha\frac{c}{p_k^{\mu_k}}m\right\}-\frac{1}{2}\right)\chi(m)
$$

and

$$
M_3(\chi)=\sum_{\substack{p_k^{\eta_k} m\le \frac{Q}{c}\\ \eta_k\ge 0}}\chi(p_k^{\eta_k})\left(\left\{\alpha\frac{c}{p_k^{\nu_k}}m\right\}-\frac{1}{2}\right)\chi(m)\,.
$$

§I-7. Proof of Theorem I-3

Let $Q > Q_o$. Let T be a sufficiently large number which will be chosen later. We write $s=\sigma+it$ with real numbers σ and t.

Let α $(0 \le \alpha \le 1)$ be irrational of type $< \psi$, where ψ satisfies the conditions given in the introduction and we suppose further that

$$\psi(t) \ll t^{\nu_o} \quad \text{with } 0 \le \nu_o < 1 .$$

We put

$$\eta = 1 + \frac{1}{\log Q}$$

and

$$\kappa = 1 - \Delta$$

with

$$\Delta = \frac{C}{(\log T)^{\frac{2}{3}} (\log\log T)^{\frac{1}{3}}} .$$

To evaluate $M_1(\chi)$, we shall consider the integral

$$\Xi_1 = -\frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \frac{L'(s, \chi\chi_{o,c})}{L(s, \chi\chi_{o,c})} Z_{c\alpha}(s, \chi) \frac{\left(\frac{Q}{c}\right)^s}{s}\, ds .$$

By Satz 3.1 on pp. 376–377 of Prachar [32], we get first that

$$\Xi_1 = M_1(\chi) + O\left(\frac{Q^\eta}{T(\eta-1)^2}\right) + O\left(\frac{Q\log Q}{T} Q^{\frac{(1+\varepsilon)\log 2}{\log\log Q}}\right) + O\left(Q^{\frac{(1+\varepsilon)\log 2}{\log\log Q}}\right).$$

On the other hand, by Cauchy's theorem, we get

$$\begin{aligned}
\Xi_1 &= -\frac{1}{2\pi\sqrt{-1}} \left(\int_{\kappa+\sqrt{-1}T}^{\eta+\sqrt{-1}T} + \int_{\kappa-\sqrt{-1}T}^{\kappa+\sqrt{-1}T} - \int_{\kappa-\sqrt{-1}T}^{\eta-\sqrt{-1}T} \right) \frac{L'(s, \chi\chi_{o,c})}{L(s, \chi\chi_{o,c})} Z_{c\alpha}(s, \chi) \frac{\left(\frac{Q}{c}\right)^s}{s}\, ds \\
&\quad + \delta(\chi) Z_{c\alpha}(1, \chi) \cdot \frac{Q}{c} \\
&= K_1 + K_2 + K_3 + \delta(\chi) Z_{c\alpha}(1, \chi) \cdot \frac{Q}{c}, \quad \text{say},
\end{aligned}$$

where we put

$$\delta(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is the principal character mod } \frac{a}{c} \\ 0 & \text{otherwise}. \end{cases}$$

Here we notice that for a fixed Dirichlet character χ mod a we have

$$\frac{L'(s, \chi)}{L(s, \chi)} \ll (\log T)^{\frac{2}{3}} (\log\log T)^{\frac{1}{3}}$$

in the region

$$\sigma \ge 1 - \frac{C}{(\log T)^{\frac{2}{3}} (\log\log T)^{\frac{1}{3}}}, \quad T_o \le |\, t \,| \le T$$

(cf. Sokolovskii [33], p. 176 of Montgomery [29] and also Tatuzawa [34] for a weaker zero free region).

In the same region, we have shown in p. 137 of Fujii [19] an upper bound of $Z_{c\alpha}(s, \chi)$ in the following form.

$$|\, Z_{c\alpha}(s, \chi) \,| \ll e^{C\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}} .$$

Using the above estimates of $\frac{L'(s,\chi)}{L(s,\chi)}$ and $Z_\alpha(s,\chi)$, we get

$$K_1, K_3 \ll (\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}} e^{C\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}} \cdot \frac{Q^\eta}{T\log Q}.$$

$$K_2 \ll Q e^{-\frac{C\log Q}{(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}}} e^{C\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}} \cdot (\log T)^{\frac{5}{3}}(\log\log T)^{\frac{1}{3}}.$$

Hence, by choosing

$$T = Q^{2^{\frac{(1+\varepsilon)\log 2}{\log\log Q}}},$$

we get

$$M_1(\chi) = \delta(\chi) Z_{c\alpha}(1,\chi)\cdot\frac{Q}{c} + O(Q\cdot e^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}}).$$

This implies

$$\begin{aligned} S_1 &= -\frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right)\delta(\chi)Z_{c\alpha}(1,\chi)\cdot\frac{Q}{c} + O(Q\cdot e^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}}) \\ &= -\frac{1}{\varphi\left(\frac{a}{c}\right)} Z_{c\alpha}(1,\chi_{o,\frac{a}{c}})\cdot\frac{Q}{c} + O(Q\cdot e^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}}).\end{aligned}$$

In the same manner as above, by considering the integral,

$$\Xi_2 = \frac{1}{2\pi\sqrt{-1}}\int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} Z_{\frac{c}{p_k^{\mu_k}}\alpha}(s,\chi)\frac{\left(\frac{Q}{c}\right)^s}{s}\,ds,$$

one obtains

$$M_2(\chi) = O(Q\cdot e^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}}).$$

Hence, we get

$$S_3 = O(Q\cdot e^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}}).$$

To evaluate $M_3(\chi)$, we notice that the generating function is

$$\sum_{\eta_k=0}^{\infty}\frac{\chi(p_k^{\eta_k})}{p_k^{\eta_k s}} Z_{\alpha\frac{c}{p_k^{\nu_k}}}(s,\chi) = \frac{1}{1-\frac{\chi(p_k)}{p_k^s}} Z_{\alpha\frac{c}{p_k^{\nu_k}}}(s,\chi).$$

In the same manner as above, by considering the integral

$$\Xi_3 = \frac{1}{2\pi\sqrt{-1}}\int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T}\frac{1}{1-\frac{\chi(p_k)}{p_k^s}} Z_{\alpha\frac{c}{p_k^{\nu_k}}}(s,\chi)\frac{\left(\frac{Q}{c}\right)^s}{s}\,ds,$$

one obtains

$$M_3(\chi) = O(Q\cdot e^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}})$$

and

$$S_4 = O(Q\cdot e^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}}).$$

Combining all evaluations, we get

$$G(\alpha) = -\frac{1}{\varphi\left(\frac{a}{c}\right)} Z_{c\alpha}(1, \chi_{o,\frac{a}{c}}) \cdot \frac{Q}{c} + O(Q \cdot e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}}) .$$

Since

$$\frac{1}{2} \sum_{\substack{n \le Q \\ n \equiv b \,(\mathrm{mod}\, a)}} \log n = \frac{1}{2a}(Q \log Q - Q) + O(\log Q) ,$$

we get our Theorem I-3.

§I-8. Proof of Theorem I-4

Let $Q > Q_o$. Let T be a sufficiently large number which will be chosen later.

We put $c = (a, b)$. Let $\alpha = \frac{p}{q}$ be a fixed reduced rational number in $0 \le \alpha \le 1$. As in p. 144 of Fujii [19], $Z_{\frac{c}{\delta}\frac{p}{q}}(s, \chi)$ is decomposed as follows.

$$\begin{aligned}
Z_{\frac{c}{\delta}\frac{p}{q}}(s, \chi) &= \sum_{n=1}^{\infty} \frac{\left\{\frac{\frac{c}{\delta}p}{q} n\right\} - \frac{1}{2}}{n^s} \chi(n) \\
&= \sum_{h=1}^{\frac{q}{d}} \left(\left\{ \frac{\frac{c}{\delta d}p}{q/d} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\,(\frac{q}{d}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j) \sum_{\substack{n=1 \\ n \equiv \rho(h,j) (\mathrm{mod}\,[\frac{q}{d}, \frac{a}{c}])}}^{\infty} \frac{1}{n^s} \\
&= \sum_{h=1}^{\frac{q}{d}} \left(\left\{ \frac{\frac{c}{\delta d}p}{q/d} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\,(\frac{q}{d}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j) \frac{1}{\left[\frac{q}{d}, \frac{a}{c}\right]^s} \zeta\left(s, \frac{\rho(h, j)}{\left[\frac{q}{d}, \frac{a}{c}\right]}\right) \\
&= \sum_{h=1}^{\frac{q}{d}} \left(\left\{ \frac{\frac{c}{\delta d}p}{q/d} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\,(\frac{q}{d}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j) \frac{1}{(\rho(h, j))^s} \\
&\quad + \left(\left[\frac{q}{d}, \frac{a}{c} \right] \right)^{-s} \sum_{h=1}^{\frac{q}{d}} \left(\left\{ \frac{\frac{c}{\delta d}p}{q/d} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \ (\mathrm{mod}\,(\frac{q}{d}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j) \\
&\quad \cdot \left(\zeta\left(s, \frac{\rho(h, j)}{\left[\frac{q}{d}, \frac{a}{c}\right]}\right) - \left(\frac{\rho(h, j)}{\left[\frac{q}{d}, \frac{a}{c}\right]} \right)^{-s} \right),
\end{aligned}$$

where we put

$$d = \left(\frac{c}{\delta}, q\right)$$

for $\delta \mid c$, $\rho(h, j)$ is the unique integer mod $\left[\frac{q}{d}, \frac{a}{c}\right]$ which is determined by the congruence conditions

$$\begin{cases} x \equiv h & \left(\bmod \frac{q}{d}\right) \\ x \equiv j & \left(\bmod \frac{a}{c}\right) \end{cases}$$

and for $0 < w \leq 1$

$$\zeta(s, w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^s} \,.$$

We put, as in the previous section, $\eta = 1 + \frac{1}{\log Q}$ and $\kappa = 1 - \Delta$ with $\Delta = \frac{C}{(\log T)^{\frac{2}{3}} (\log\log T)^{\frac{1}{3}}}$. We have the estimate

$$\zeta\left(s, \frac{\rho(h, j)}{\left[\frac{q}{d}, \frac{a}{c}\right]}\right) - \left(\frac{\rho(h, j)}{\left[\frac{q}{d}, \frac{a}{c}\right]}\right)^{-s} \ll e^{C\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}}$$

in the region $\sigma \geq 1 - \Delta$ and $2 \leq |\, t \,| \ll T$ (cf. p. 145 of Fujii [19]).

We shall evaluate first $M_1(\chi)$. We consider the integral

$$\begin{aligned}
\Xi_1 &= -\frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \frac{L'(s, \chi\chi_{o,c})}{L(s, \chi\chi_{o,c})} Z_{c\frac{p}{q}}(s, \chi) \frac{\left(\frac{Q}{c}\right)^s}{s}\, ds \\
&= -\sum_{h=1}^{\frac{q}{d_1}} \left(\left\{\frac{\frac{c}{d_1}p}{q/d_1} h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j \equiv h \ (\bmod\, (\frac{q}{d_1}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j) \\
&\qquad \cdot \frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \frac{L'(s, \chi\chi_{o,c})}{L(s, \chi\chi_{o,c})} \frac{\left(\frac{Q}{\rho(h,j)c}\right)^s}{s}\, ds \\
&\quad - \sum_{h=1}^{\frac{q}{d_1}} \left(\left\{\frac{\frac{c}{d_1}p}{q/d_1} h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j \equiv h \ (\bmod\, (\frac{q}{d_1}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j) \\
&\qquad \cdot \frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \frac{L'(s, \chi\chi_{o,c})}{L(s, \chi\chi_{o,c})} \left(\zeta(s, \frac{\rho(h, j)}{\left[\frac{q}{d_1}, \frac{a}{c}\right]}) - \left(\frac{\rho(h, j)}{\left[\frac{q}{d_1}, \frac{a}{c}\right]}\right)^{-s}\right) \frac{\left(\frac{Q}{[\frac{q}{d_1}, \frac{a}{c}]c}\right)^s}{s}\, ds \\
&= K_4 + K_5 \,, \quad \text{say}\,,
\end{aligned}$$

where we put $d_1 = (c, q)$. As in the previous section, we get, by Cauchy's theorem,

$$\begin{aligned}
K_4 = &\sum_{h=1}^{\frac{q}{d_1}} \left(\left\{\frac{\frac{c}{d_1}p}{q/d_1} h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j \equiv h \ (\bmod\, (\frac{q}{d_1}, \frac{a}{c}))}}^{\frac{a}{c}} \chi(j)\delta(\chi) \frac{Q}{\rho(h, j)c} \\
&+ O\left(\frac{Q}{T \log Q} (\log T)^{\frac{2}{3}} (\log\log T)^{\frac{1}{3}}\right)
\end{aligned}$$

$$+O(Qe^{-\frac{C\log Q}{(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}}}(\log T)^{\frac{5}{3}}(\log\log T)^{\frac{1}{3}}).$$

By Cauchy's theorem, we get also

$$\begin{aligned}
K_5 = &-\sum_{h=1}^{\frac{q}{d_1}}\left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h \pmod{(\frac{q}{d_1},\frac{a}{c})}}}^{\frac{a}{c}}\chi(j)\\
&\cdot\frac{1}{2\pi\sqrt{-1}}\left(\int_{\kappa+\sqrt{-1}T}^{\eta+\sqrt{-1}T}+\int_{\kappa-\sqrt{-1}T}^{\kappa+\sqrt{-1}T}-\int_{\kappa-\sqrt{-1}T}^{\eta-\sqrt{-1}T}\right)\frac{L'(s,\chi\chi_{o,c})}{L(s,\chi\chi_{o,c})}\\
&\cdot\left(\zeta\left(s,\frac{\rho(h,j)}{\left[\frac{q}{d_1},\frac{a}{c}\right]}\right)-\left(\frac{\rho(h,j)}{\left[\frac{q}{d_1},\frac{a}{c}\right]}\right)^{-s}\right)\frac{\left(\frac{Q}{[\frac{q}{d_1},\frac{a}{c}]c}\right)^s}{s}\,ds\\
&+\sum_{h=1}^{\frac{q}{d_1}}\left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h \pmod{(\frac{q}{d_1},\frac{a}{c})}}}^{\frac{a}{c}}\chi(j)\\
&\cdot\delta(\chi)\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}\left(-\sum_{p|a}\frac{\log p}{p-1}-\gamma_o+\gamma_o\left(\frac{\rho(h,j)}{\left[\frac{q}{d_1},\frac{a}{c}\right]}\right)+\log\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}-1\right)\\
&-\sum_{h=1}^{\frac{q}{d_1}}\left(\left\{\frac{\frac{c}{d}p}{q/d_1}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h \pmod{(\frac{q}{d_1},\frac{a}{c})}}}^{\frac{a}{c}}\chi(j)\\
&\cdot(1-\delta(\chi))\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}\frac{L'(1,\chi\chi_{o,c})}{L(1,\chi\chi_{o,c})}\\
= &K_6+K_7+K_8+\sum_{h=1}^{\frac{q}{d_1}}\left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h \pmod{(\frac{q}{d_1},\frac{a}{c})}}}^{\frac{a}{c}}\chi(j)\\
&\cdot\delta(\chi)\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}\left(-\sum_{p|a}\frac{\log p}{p-1}-\gamma_o+\gamma_o\left(\frac{\rho(h,j)}{\left[\frac{q}{d_1},\frac{a}{c}\right]}\right)+\log\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}-1\right)\\
&-\sum_{h=1}^{\frac{q}{d_1}}\left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h \pmod{(\frac{q}{d_1},\frac{a}{c})}}}^{\frac{a}{c}}\chi(j)(1-\delta(\chi))\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}\frac{L'(1,\chi\chi_{o,c})}{L(1,\chi\chi_{o,c})},\quad\text{say},
\end{aligned}$$

where for the principal character $\chi_{o,a}$, we have

$$\frac{L'(s,\chi_{o,a})}{L(s,\chi_{o,a})}=-\frac{1}{s-1}+\sum_{p|a}\frac{\log p}{p-1}+\gamma_o-C_1(\chi_{o.a})(s-1)+\cdots$$

and we put

$$\zeta(s,,w) - w^{-s} = \frac{1}{s-1} + \gamma_o(w) + D_1(s-1) + \cdots .$$

It is easily seen (cf. p. 30 of Montgomery-Vaughan [30]) that

$$\gamma_0(w) = -\log(1+w) + \frac{1}{2}\frac{1}{1+w} - \int_1^\infty \frac{\{x\} - \frac{1}{2}}{(x+w)^2}\,dx$$

and that

$$\gamma_o(w) = -\frac{\Gamma'}{\Gamma}(w) - \frac{1}{w}.$$

Using the above estimates on $\frac{L'(s,\chi\chi_{o,c})}{L(s,\chi\chi_{o,c})}$ and $\zeta\left(s, \frac{\rho(h,j)}{[\frac{q}{d_1},\frac{a}{c}]}\right) - \left(\frac{\rho(h,j)}{[\frac{q}{d_1},\frac{a}{c}]}\right)^{-s}$, we get

$$K_6, K_8 \ll (\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}} e^{C\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}} \cdot \frac{Q^\eta}{T\log Q}$$

and

$$K_7 \ll Qe^{-\frac{C\log Q}{(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}}} e^{C\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}} \cdot (\log T)^{\frac{5}{3}}(\log\log T)^{\frac{1}{3}}.$$

Hence, we get

$$\begin{aligned}
M_1(\chi) = &\sum_{h=1}^{\frac{q}{d_1}} \left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j\equiv h \pmod{(\frac{q}{d_1},\frac{a}{c})}}}^{\frac{a}{c}} \chi(j) \\
&\cdot \delta(\chi)\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}\left(-\sum_{p|a}\frac{\log p}{p-1} - \gamma_o + \gamma_o\left(\frac{\rho(h,j)}{\left[\frac{q}{d_1},\frac{a}{c}\right]}\right) + \log\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c} - 1\right) \\
&- \sum_{h=1}^{\frac{q}{d_1}} \left(\left\{\frac{\frac{c}{\delta d}p}{q/d_1}h\right\} - \frac{1}{2}\right) \sum_{\substack{j=1 \\ j\equiv h \pmod{(\frac{q}{d_1},\frac{a}{c})}}}^{\frac{a}{c}} \chi(j) \\
&\cdot (1-\delta(\chi))\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}\frac{L'(1,\chi\chi_{o,c})}{L(1,\chi\chi_{o,c})} \\
&+ O\left((\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}} e^{C\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}} \cdot \frac{Q^\eta}{T\log Q}\right) \\
&+ O(Qe^{-\frac{C\log Q}{(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}}} e^{C\left(\frac{\log T}{\log\log T}\right)^{\frac{1}{3}}} \cdot (\log T)^{\frac{5}{3}}(\log\log T)^{\frac{1}{3}}) \\
&+ O\left(\frac{Q^\eta}{T(\eta-1)^2}\right) + O\left(\frac{Q\log Q}{T}Q^{\frac{(1+\varepsilon)\log 2}{\log\log Q}}\right) + O\left(Q^{\frac{(1+\varepsilon)\log 2}{\log\log Q}}\right).
\end{aligned}$$

Thus by taking

$$T = Q^{2\frac{(1+\varepsilon)\log 2}{\log\log Q}},$$

we get

$$S_1 = -\frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{h=1}^{\frac{q}{d_1}} \left(\left\{ \frac{\frac{c}{d_1} p}{q/d_1} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_1}, \frac{a}{c})}}}^{\frac{a}{c}} \chi_{o,\frac{a}{c}}(j)$$

$$\cdot \frac{Q}{\left[\frac{q}{d_1}, \frac{a}{c}\right] c} \left(-\sum_{p|a} \frac{\log p}{p-1} - \gamma_o + \gamma_o \left(\frac{\rho(h,j)}{\left[\frac{q}{d_1}, \frac{a}{c}\right]} \right) + \log \frac{Q}{\left[\frac{q}{d_1}, \frac{a}{c}\right] c} - 1 \right)$$

$$+ \frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\chi \bmod \frac{a}{c}, \chi \neq \chi_{o,\frac{a}{c}}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{h=1}^{\frac{q}{d_1}} \left(\left\{ \frac{\frac{c}{d_1} p}{q/d_1} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_1}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j)$$

$$\cdot \frac{Q}{\left[\frac{q}{d_1}, \frac{a}{c}\right] c} \frac{L'(1, \chi \chi_{o,c})}{L(1, \chi \chi_{o,c})} + O(Q e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}}).$$

To evaluate $M_2(\chi)$, we consider the integral

$$\Xi_2 = \frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} Z_{\frac{c}{p_k^{\mu_k}}\alpha}(s, \chi) \frac{\left(\frac{Q}{c}\right)^s}{s} \, ds$$

and get in the same manner as above

$$\Xi_2 = \sum_{h=1}^{\frac{q}{d_2}} \left(\left\{ \frac{\frac{c}{p_k^{\mu_k} d_2} p}{q/d_2} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_2}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j) \frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \frac{\left(\frac{Q}{\rho(h,j)c}\right)^s}{s} \, ds$$

$$+ \sum_{h=1}^{\frac{q}{d_2}} \left(\left\{ \frac{\frac{c}{p_k^{\mu_k} d_2} p}{q/d_2} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_2}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j)$$

$$\cdot \frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \left(\zeta\left(s, \frac{\rho(h,j)}{\left[\frac{q}{d_2}, \frac{a}{c}\right]}\right) - \left(\frac{\rho(h,j)}{\left[\frac{q}{d_2}, \frac{a}{c}\right]}\right)^{-s}\right) \frac{\left(\frac{Q}{\left[\frac{q}{d_2}, \frac{a}{c}\right] c}\right)^s}{s} \, ds$$

$$= \sum_{h=1}^{\frac{q}{d_2}} \left(\left\{ \frac{\frac{c}{p_k^{\mu_k} d_2} p}{q/d_2} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_2}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j) \frac{Q}{\left[\frac{q}{d_2}, \frac{a}{c}\right] c} + O(Q e^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}})$$

and

$$S_3 = -(1 - \delta_1(c)) \frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{k=1}^{r} \delta_2(\nu_k) \log p_k$$

$$\cdot \sum_{\mu_k=1}^{\nu_k-1} \sum_{h=1}^{\frac{q}{d_2}} \left(\left\{ \frac{\frac{c}{p_k^{\mu_k} d_2} p}{q/d_2} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_2}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j) \frac{Q}{\left[\frac{q}{d_2}, \frac{a}{c}\right] c}$$

$$+O(Qe^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}}),$$

where we put $d_2 = \left(q, \frac{c}{p_k^{\mu_k}}\right)$

To evaluate $M_3(\chi)$, we consider the integral

$$\Xi_3 = \frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \frac{1}{1-\frac{\chi(p_k)}{p_k^s}} Z_{\frac{c}{p_k^{\nu_k}}\alpha}(s, \chi) \frac{\left(\frac{Q}{c}\right)^s}{s}\, ds$$

and get

$$\Xi_3 = \sum_{h=1}^{\frac{q}{d_3}} \left(\left\{ \frac{\frac{c}{p_k^{\nu_k} d_3} p}{q/d_3} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_3}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j)$$

$$\cdot \frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \frac{1}{1-\frac{\chi(p_k)}{p_k^s}} \frac{\left(\frac{Q}{\rho(h,j)c}\right)^s}{s}\, ds$$

$$+\sum_{h=1}^{\frac{q}{d_3}} \left(\left\{ \frac{\frac{c}{p_k^{\nu_k} d_3} p}{q/d_3} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_3}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j)$$

$$\cdot \frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}T}^{\eta+\sqrt{-1}T} \frac{1}{1-\frac{\chi(p_k)}{p_k^s}} \left(\zeta\left(s, \frac{\rho(h,j)}{\left[\frac{q}{d_3}, \frac{a}{c}\right]}\right) - \left(\frac{\rho(h,j)}{\left[\frac{q}{d_3}, \frac{a}{c}\right]}\right)^{-s} \right) \frac{\left(\frac{Q}{[\frac{q}{d_3}, \frac{a}{c}]c}\right)^s}{s}\, ds$$

$$= \sum_{h=1}^{\frac{q}{d_3}} \left(\left\{ \frac{\frac{c}{p_k^{\nu_k} d_3} p}{q/d_3} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_3}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j) \frac{Q}{\left[\frac{q}{d_3}, \frac{a}{c}\right] c} \frac{1}{1-\frac{\chi(p_k)}{p_k}}$$

$$+O(Qe^{-C(\log Q \cdot \log\log Q)^{\frac{1}{3}}})$$

and

$$S_4 = -(1-\delta_1(c)) \frac{1}{\varphi\left(\frac{a}{c}\right)} \sum_{\chi \bmod \frac{a}{c}} \bar{\chi}\left(\frac{b}{c}\right) \sum_{k=1}^{r} \log p_k \frac{1}{1-\frac{\chi(p_k)}{p_k}}$$

$$\cdot \sum_{h=1}^{\frac{q}{d_3}} \left(\left\{ \frac{\frac{c}{p_k^{\nu_k} d_3} p}{q/d_3} h \right\} - \frac{1}{2} \right) \sum_{\substack{j=1 \\ j \equiv h \pmod{(\frac{q}{d_3}, \frac{a}{c})}}}^{\frac{a}{c}} \chi(j) \frac{Q}{\left[\frac{q}{d_3}, \frac{a}{c}\right] c}$$

$$+O(Qe^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}}),$$

where we put $d_3=(q,\frac{c}{p_k^{\nu_k}})$.

Consequently, we get

$$G(\alpha)=-\frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{h=1}^{\frac{q}{d_1}}\left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h\ (\mathrm{mod}\,(\frac{q}{d_1},\frac{a}{c}))}}^{\frac{a}{c}}\chi_{o,\frac{a}{c}}(j)$$

$$\cdot\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}\left(-\sum_{p|a}\frac{\log p}{p-1}-\gamma_o-\frac{\Gamma'}{\Gamma}\left(\frac{\rho(h,j)}{\left[\frac{q}{d_1},\frac{a}{c}\right]}\right)-\frac{\left[\frac{q}{d_1},\frac{a}{c}\right]}{\rho(h,j)}+\log\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}-1\right)$$

$$+\frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\chi\ \mathrm{mod}\ \frac{a}{c},\chi\neq\chi_{o,\frac{a}{c}}}\bar{\chi}\left(\frac{b}{c}\right)\sum_{h=1}^{\frac{q}{d_1}}\left(\left\{\frac{\frac{c}{d_1}p}{q/d_1}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h\ (\mathrm{mod}\,(\frac{q}{d_1},\frac{a}{c}))}}^{\frac{a}{c}}\chi(j)$$

$$\cdot\frac{Q}{\left[\frac{q}{d_1},\frac{a}{c}\right]c}\frac{L'(1,\chi\chi_{o,c})}{L(1,\chi\chi_{o,c})}$$

$$-(1-\delta_1(c))\frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\chi\ \mathrm{mod}\ \frac{a}{c}}\bar{\chi}\left(\frac{b}{c}\right)\sum_{k=1}^{r}\delta_2(\nu_k)\log p_k$$

$$\cdot\sum_{\mu_k=1}^{\nu_k-1}\sum_{h=1}^{\frac{q}{d_2}}\left(\left\{\frac{\frac{c}{p_k^{\mu_k}d_2}p}{q/d_2}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h\ (\mathrm{mod}\,(\frac{q}{d_2},\frac{a}{c}))}}^{\frac{a}{c}}\chi(j)\frac{Q}{\left[\frac{q}{d_2},\frac{a}{c}\right]c}$$

$$-(1-\delta_1(c))\frac{1}{\varphi\left(\frac{a}{c}\right)}\sum_{\chi\ \mathrm{mod}\ \frac{a}{c}}\bar{\chi}\left(\frac{b}{c}\right)\sum_{k=1}^{r}\log p_k\frac{1}{1-\frac{\chi(p_k)}{p_k}}$$

$$\cdot\sum_{h=1}^{\frac{q}{d_3}}\left(\left\{\frac{\frac{c}{p_k^{\nu_k}d_3}p}{q/d_3}h\right\}-\frac{1}{2}\right)\sum_{\substack{j=1\\ j\equiv h\ (\mathrm{mod}\,(\frac{q}{d_3},\frac{a}{c}))}}^{\frac{a}{c}}\chi(j)\frac{Q}{\left[\frac{q}{d_3},\frac{a}{c}\right]c}$$

$$+O(Qe^{-C(\log Q\cdot\log\log Q)^{\frac{1}{3}}}).$$

This proves our Theorem I-4.

Part II. Some Applications of Baker's Theorem to a Study of the Values of Hecke L-functions at $s = 1$.

§II-1. Introduction

In Part II, we shall show, as an immediate consequence of Baker's theory and Murty-Saradha [31], that some special values related with Hecke L-functions are transcendental. We shall also give some open problems as conjectures.

Baker's theory (cf. Baker [2]) is strong and it has some applications even in the theory of the Riemann zeta function (cf. Fujii [16]). We start with recalling some of the author's works concerning the applications of Baker's theory to the theory of the distribution of the zeros of the Riemann zeta function $\zeta(s)$. In Fujii [14] [15], we have shown under the Riemann Hypothesis that for any positive α and $T > T_o$

$$\sum_{\gamma \leq T} \left(\left\{ \alpha \frac{\gamma}{2\pi} \right\}^2 - \left\{ \alpha \frac{\gamma}{2\pi} \right\} + \frac{1}{6} \right) \ll T \,,$$

where γ runs over the positive imaginary parts of the zeros of $\zeta(s)$. Concerning this sum, we have shown the following , by applying Baker's theory, under the Riemann Hypothesis.

(i) If either α or e^{α} is algebraic, then for any $T > T_o$ we have

$$\sum_{\gamma \leq T} \left(\left\{ \alpha \frac{\gamma}{2\pi} \right\}^2 - \left\{ \alpha \frac{\gamma}{2\pi} \right\} + \frac{1}{6} \right)$$
$$= -\frac{T}{2\pi^3} \frac{\Lambda(e^{G\alpha})}{G^2} \mathrm{Li}_2(e^{\frac{G}{2}\alpha}) + O\left(\frac{T}{\log T} (\log\log T)^2 \right),$$

where we put

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

and G is either the minimum integer $n(\geq 1)$ such that $e^{n\alpha}$ is a prime power, or $\frac{1}{\alpha}$ if such n does not exist.

Thus we see that the upper bound on the above sum for general α is best possible. We recall second the following result which has been shown also under the Riemann Hypothesis by applying Baker's theory (cf. Fujii [13]).

(ii) For any algebraic number $X(> 1)$,

$$\sum_{\gamma > 0} \frac{\sin(\gamma \log X)}{\gamma} - \frac{1}{2}\mathrm{Arctan}\frac{1}{\sqrt{X}} + \frac{1}{4}\gamma_o + \frac{1}{4}\log\pi$$

is a transcendental number.

In Part II, we are concerned with the values of Hecke L-functions at $s = 1$. We may start with proposing the following open problem as a conjecture.

CONJECTURE 1. *For any irrational* $\alpha > 0$, *the number*

$$\sum_{n=1}^{\infty} \frac{\{\alpha n\} - \frac{1}{2}}{n}$$

is transcendental.

We can consider the same problem for a rational α. For any rational $0 < \frac{p}{q} < 1$, $(p, q) = 1$, the series

$$\sum_{n=1}^{\infty} \frac{\left\{\frac{p}{q}n\right\} - \frac{1}{2}}{n^s}$$

is convergent for $\Re(s) > 1$ and can be written as follows.

$$\begin{aligned} Z_{\frac{p}{q}}(s) &= \sum_{n=1}^{\infty} \frac{\left\{\frac{p}{q}n\right\} - \frac{1}{2}}{n^s} = \sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) \sum_{m=0}^{\infty} \frac{1}{(b+mq)^s} \\ &= q^{-s} \sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) \zeta\left(s, \frac{b}{q}\right), \end{aligned}$$

where $\zeta(s, \omega) = \sum_{m=0}^{\infty} \frac{1}{(m+\omega)^s}$ is the Hurwitz zeta function for any positive number $\omega \leq 1$ as introduced in the section 8. Hence, at $s = 1$, we have

$$\begin{aligned} Z_{\frac{p}{q}}(s) &= \sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right)\left(\frac{1}{q} - \frac{\log q}{q}(s-1) + \cdots\right) \cdot \left(\frac{1}{s-1} - \psi(\frac{b}{q}) + O(s-1)\right) \\ &= \frac{1}{q} \sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) \frac{1}{s-1} - \frac{\log q}{q} \sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) \\ &\quad - \frac{1}{q} \sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) \psi\left(\frac{b}{q}\right) + O(s-1) \\ &= -\frac{1}{2q} \frac{1}{s-1} + \frac{1}{2} \frac{\log q}{q} - \frac{1}{q} \sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) \psi\left(\frac{b}{q}\right) + O(s-1), \end{aligned}$$

where we put

$$\psi(x) = \frac{\Gamma'}{\Gamma}(x)$$

and we notice that

$$\sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) = -\frac{1}{2}.$$

We denote the constant term by $C(\frac{p}{q})$. Namely, we put

$$C\left(\frac{p}{q}\right) = \frac{1}{2} \frac{\log q}{q} - \frac{1}{q} \sum_{b=1}^{q} \left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) \psi\left(\frac{b}{q}\right).$$

We may state the following natural problem as a conjecture.

CONJECTURE 2. *$C\left(\frac{p}{q}\right)$ is transcendental for any positive rational $\frac{p}{q} \leq 1$ with $(p,q)=1$.*

We notice that $-\psi(1)=\gamma_o$ and that when $q=2$, we have

$$C\left(\frac{1}{2}\right)=\frac{1}{2}\frac{\log 2}{2}-\frac{1}{2}\sum_{b=1}^{2}\left(\left\{\frac{1}{2}b\right\}-\frac{1}{2}\right)\psi\left(\frac{b}{2}\right)$$
$$=\frac{\log 2}{4}-\frac{1}{4}\gamma_o\,.$$

Hence by Lindemann's theorem, we see that $C(\frac{1}{2})+\frac{1}{4}\gamma_o$ is transcendental. More generally, we have the following theorem.

THEOREM II-1. (i) *For any integer $q \geq 3$,*

$$C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}$$

is transcendental.

(ii) *For any integer $q \geq 2$,*

$$C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o$$

is transcendental.

On the other hand, we can show the following theorem.

THEOREM II-2. *Suppose that q is a prime number≥ 3. Then for all integer p with $1 \leq p \leq q-1$,*

$$C\left(\frac{p}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}$$

is transcendental.

We have introduced and studied a more general Hecke L-function.

$$Z_\alpha(s,\chi)=\sum_{n=1}^{\infty}\frac{\{\alpha n\}-\frac{1}{2}}{n^s}\chi(n)\,,$$

where χ is a Dirichlet character. We may state the following problem as a conjecture.

CONJECTURE 3. *For any real $\alpha>0$ and for any Dirichlet character χ, the number*

$$\sum_{n=1}^{\infty}\frac{\{\alpha n\}-\frac{1}{2}}{n}\chi(n)$$

is transcendental, unless α is rational and χ is the principal character.

Concerning Conjecture 3, we shall prove the following theorem.

THEOREM II-2. *Let q be an integer ≥ 3. Suppose that χ is a non-principal character mod q, $\chi(-1)=1$ and $(q,\varphi(q))=1$. Then for all integer p with $1 \leq p \leq q-1$ and*

$(p, q) = 1$,

$$\sum_{n=1}^{\infty} \frac{\left\{\frac{p}{q}n\right\} - \frac{1}{2}}{n} \chi(n)$$

is transcendental.

On this occasion, we notice that the main part of Part II has been obtained while the author has visited Harish-Chandra Research Institute at Allahabad in 2007. The author wishes to express his thanks to Professor Sukmar Adhikari for his kind hospitality, to Professor Ram Murty for his nice lecture at the institute on that occasion, which inspired Part II, and also to Professor R.Balasubramanian for his kind hospitality during my visit to Institute of Mathematical Science at Chennai in the same year.

§II-2. Proof of Theorem II-1

It is a well-known formula of Gauss (cf. p. 528 of Montgomery-Vaughan [30]) that

$$-\psi\left(\frac{b}{q}\right) = \gamma_o + \log q - \sum_{c=1}^{q-1} e^{-2\pi i \frac{cb}{q}} \log(1 - e^{2\pi i \frac{c}{q}}).$$

Inserting this directly, we get

$$\begin{aligned}
&C\left(\frac{p}{q}\right) + \frac{1}{2q}\gamma_o - \frac{\log q}{2q} \\
&\quad = \frac{1}{q}\sum_{b=1}^{q-1}\left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right)\left(\gamma_o + \log q - \sum_{c=1}^{q-1} e^{-2\pi i \frac{cb}{q}} \log(1 - e^{2\pi i \frac{c}{q}})\right) \\
&\quad = (\gamma_o + \log q)\frac{1}{q}\sum_{b=1}^{q-1}\left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right) \\
&\quad -\frac{1}{q}\sum_{c=1}^{q-1}\left(\sum_{b=1}^{q-1}\left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right)e^{-2\pi i \frac{cb}{q}}\right)\log(1 - e^{2\pi i \frac{c}{q}}) \\
&\quad = -\frac{1}{q}\sum_{c=1}^{q-1}\left(\sum_{b=1}^{q-1}\left(\left\{\frac{p}{q}b\right\} - \frac{1}{2}\right)e^{-2\pi i \frac{cb}{q}}\right)\log(1 - e^{2\pi i \frac{c}{q}}).
\end{aligned}$$

Consequently, Baker's theory implies that $C\left(\frac{p}{q}\right) + \frac{1}{2q}\gamma_o - \frac{\log q}{2q}$ is transcendental if $C\left(\frac{p}{q}\right) + \frac{1}{2q}\gamma_o - \frac{\log q}{2q} \neq 0$. And that $C\left(\frac{p}{q}\right) + \frac{1}{2q}\gamma_o$ is transcendental if $C\left(\frac{p}{q}\right) + \frac{1}{2q}\gamma_o \neq 0$.

In the rest of this section, we assume that $p = 1$. We shall prove first that if $q \geq 3$, then $C\left(\frac{1}{q}\right) + \frac{1}{2q}\gamma_o - \frac{\log q}{2q}$ is transcendental, namely,

$$C\left(\frac{1}{q}\right) + \frac{1}{2q}\gamma_o - \frac{\log q}{2q} \neq 0.$$

For this purpose we shall use the following expression of $\psi(x)$ (cf. C.10 on p. 522 of Montgomery and Vaughan [30])).

$$-\psi(x)=\gamma_o+\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{n+x}-\frac{1}{n}\right).$$

Then we have

$$\begin{aligned}
C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}&=\frac{1}{q}\sum_{b=1}^{q-1}\left(\frac{b}{q}-\frac{1}{2}\right)\left(\gamma_o+\frac{1}{\frac{b}{q}}-\frac{b}{q}\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{b}{q}\right)}\right)\\
&=\frac{1}{q}\sum_{b=1}^{q-1}\left(\frac{b}{q}-\frac{1}{2}\right)\left(\frac{1}{\frac{b}{q}}-\frac{b}{q}\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{b}{q}\right)}\right)\\
&=\frac{1}{q}\sum_{1\le b<\frac{q}{2}}\left(\frac{b}{q}-\frac{1}{2}\right)\left(\frac{1}{\frac{b}{q}}-\frac{b}{q}\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{b}{q}\right)}\right)\\
&\quad+\frac{1}{q}\sum_{1\le b\le\frac{q}{2}}\left(\frac{q-b}{q}-\frac{1}{2}\right)\left(\frac{1}{\frac{q-b}{q}}-\frac{q-b}{q}\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{q-b}{q}\right)}\right)\\
&=-\frac{1}{q}\sum_{1\le b<\frac{q}{2}}\left(\frac{1}{2}-\frac{b}{q}\right)\left(\frac{1}{\frac{b}{q}}-\frac{b}{q}\sum_{n=1}^{\infty}\frac{1}{n\left(n+\frac{b}{q}\right)}\right.\\
&\quad\left.-\frac{1}{1-\frac{b}{q}}+\left(1-\frac{b}{q}\right)\sum_{n=1}^{\infty}\frac{1}{n\left(n+1-\frac{b}{q}\right)}\right).
\end{aligned}$$

The last sum can be simplified further and we get

$$C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}=-\frac{2}{q}\sum_{1\le b<\frac{q}{2}}\left(\frac{1}{2}-\frac{b}{q}\right)^2\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{b}{q}\right)\left(n+1-\frac{b}{q}\right)}.$$

Hence, if $q\ge 3$, then it is clear that

$$C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}<0.$$

In particular,

$$C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}\neq 0.$$

Hence, if $q\neq 2$, then $C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}$ is transcendental.

We shall next prove that if $q\ge 3$, then $C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o$ is transcendental, namely,

$$C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o\neq 0.$$

We suppose first that $q \geq 5$. In this case, we have

$$2\sum_{1\leq b<\frac{q}{2}}\left(\frac{1}{2}-\frac{b}{q}\right)^2\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{b}{q}\right)\left(n+1-\frac{b}{q}\right)}$$

$$\geq 2\left(\frac{1}{2}-\frac{1}{q}\right)^2\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{q}\right)\left(n+1-\frac{1}{q}\right)} > 2\left(\frac{q-2}{2q}\right)^2\frac{q^2}{q-1}\,.$$

If $q \geq 5$, then we have

$$2\left(\frac{q-2}{2q}\right)^2\frac{q^2}{q-1} > \frac{\log q}{2}\,.$$

Hence if $q \geq 5$, then

$$C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o < 0\,.$$

In particular, we have

$$C\left(\frac{1}{q}\right)+\frac{1}{2q}\gamma_o \neq 0\,.$$

When $q = 4$, then

$$2\sum_{1\leq b<2}\left(\frac{1}{2}-\frac{b}{4}\right)^2\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{b}{4}\right)\left(n+1-\frac{b}{4}\right)}$$

$$=2\left(\frac{1}{2}-\frac{1}{4}\right)^2\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{4}\right)\left(n+\frac{3}{4}\right)}$$

$$=2\sum_{n=0}^{\infty}\frac{1}{(4n+1)(4n+3)} > 2\cdot\left(\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}\right) > 0.7238 > \frac{\log 4}{2}\,.$$

Hence when $q = 4$, then

$$C\left(\frac{1}{4}\right)+\frac{1}{8}\gamma_o < 0\,.$$

In particular, we have

$$C\left(\frac{1}{4}\right)+\frac{1}{8}\gamma_o \neq 0\,.$$

When $q = 3$, then

$$2\sum_{1\leq b<\frac{3}{2}}\left(\frac{1}{2}-\frac{b}{3}\right)^2\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{b}{3}\right)\left(n+1-\frac{b}{3}\right)}$$

$$=2\left(\frac{1}{2}-\frac{1}{3}\right)^2\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{3}\right)\left(n+\frac{2}{3}\right)}$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1\cdot 2}+\frac{1}{4\cdot 5}+\frac{1}{7\cdot 8}+\frac{1}{10\cdot 11}+\frac{1}{13\cdot 14}+\cdots\right)$$

$$< \frac{1}{2} \cdot \left(\frac{1}{1 \cdot 2} + \frac{1}{4 \cdot 5} + \frac{1}{7 \cdot 8} + \frac{1}{10 \cdot 11} + \sum_{n=13}^{\infty} \frac{1}{n^2} \right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{1 \cdot 2} + \frac{1}{4 \cdot 5} + \frac{1}{7 \cdot 8} + \frac{1}{10 \cdot 11} + \frac{\pi^2}{6} - \sum_{n=1}^{12} \frac{1}{n^2} \right)$$

$$< 0.3285 < 0.549 < \frac{\log 3}{2} .$$

Hence, when $q = 3$, then

$$C\left(\frac{1}{3}\right) + \frac{1}{6}\gamma_o > 0 .$$

In particular, we have

$$C\left(\frac{1}{3}\right) + \frac{1}{6}\gamma_o \neq 0 .$$

These prove Theorem II-1.

§II-3. Proof of Theorem II-2

We suppose that q is a prime number in this section.

$$Z_{\frac{p}{q}}(s) = q^{-s} \sum_{b=1}^{q} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \zeta\left(s, \frac{b}{q}\right)$$

$$= q^{-s}(-\frac{1}{2})\zeta(s, 1) + q^{-s} \sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \zeta\left(s, \frac{b}{q}\right)$$

$$= q^{-s}\left(-\frac{1}{2}\right)\zeta(s, 1) + \frac{1}{\varphi(q)} \sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \sum_{\chi \bmod q} \bar{\chi}(b) L(s, \chi)$$

$$= q^{-s}\left(-\frac{1}{2}\right)\zeta(s, 1) + \frac{1}{\varphi(q)} \sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \bar{\chi}_o(b) L(s, \chi_o)$$

$$+ \frac{1}{\varphi(q)} \sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \sum_{\chi \neq \chi_o \bmod q} \bar{\chi}(b) L(s, \chi) ,$$

where χ_o is the principal character mod q. Hence, we get at $s = 1$

$$Z_{\frac{p}{q}}(s) = -\frac{1}{2q}\frac{1}{s-1} - \frac{1}{2q}\gamma_o + \frac{\log q}{2q}$$

$$+ \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_o \bmod q} \left(\sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \bar{\chi}(b) \right) L(1, \chi) + O(s-1) + \cdots .$$

Thus we get

$$C\left(\frac{p}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}=\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_o \bmod q}\left(\sum_{b=1}^{q-1}\left(\left\{\frac{p}{q}b\right\}-\frac{1}{2}\right)\bar{\chi}(b)\right)L(1,\chi)\,.$$

Here we notice the following result due to Murty-Saradha [31] (cf. Corollary 5 on p. 301 of [31]).

LEMMA C. *Suppose that $(q,\varphi(q))=1$. Let $K=Q(\xi)$, where ξ is a primitive $\varphi(q)$th root of unity. Then, any combination*

$$\sum_{\chi\neq\chi_o} a_\chi L(1,\chi)$$

with $a_\chi\in K$, not all zero, is transcendental, where χ runs over all Dirichlet characters mod q and χ_o is the principal character mod q.

Now we have

$$\begin{aligned}\sum_{b=1}^{q-1}&\left(\left\{\frac{p}{q}b\right\}-\frac{1}{2}\right)\bar{\chi}(b)\\ &=\chi(p)\sum_{j=1}^{q-1}\left(\frac{j}{q}-\frac{1}{2}\right)\bar{\chi}(j)\\ &=\frac{\chi(p)}{q}\sum_{j=1}^{q-1}j\bar{\chi}(j)-\frac{1}{2}\chi(p)\sum_{j=1}^{q-1}\bar{\chi}(j)\\ &=\frac{\chi(p)}{q}\sum_{j=1}^{q-1}j\bar{\chi}(j)\,.\end{aligned}$$

If $\chi(-1)=-1$, then $\sum_{j=1}^{q-1}j\bar{\chi}(j)\neq 0$. Thus we get

$$C\left(\frac{p}{q}\right)+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}=\frac{1}{\varphi(q)q}\sum_{\substack{\chi\neq\chi_o \bmod q\\ \chi(-1)=-1}}(\chi(p)\sum_{j=1}^{q-1}j\bar{\chi}(j))L(1,\chi)\,.$$

Since $(q,\varphi(q))=1$ in the present case, we can apply Lemma C and conclude that $C(\frac{p}{q})+\frac{1}{2q}\gamma_o-\frac{\log q}{2q}$ is transcendental.

§II-4. Proof of Theorem II-3

Let χ be a non-principal Dirichlet character mod q.

$$Z_{\frac{p}{q}}(s,\chi)=q^{-s}\sum_{b=1}^{q-1}\left(\left\{\frac{p}{q}b\right\}-\frac{1}{2}\right)\chi(b)\sum_{m=0}^{\infty}\frac{1}{(m+\frac{b}{q})^s}$$

$$
\begin{aligned}
&= \frac{1}{\varphi(q)} \sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \chi(b) \sum_{\psi \bmod q} \bar{\psi}(b) L(s, \psi) \\
&= \frac{1}{\varphi(q)} \sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \chi(b) \bar{\psi}_o(b) L(s, \psi_o) \\
&\quad + \frac{1}{\varphi(q)} \sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \chi(b) \sum_{\psi \neq \psi_o \bmod q} \bar{\psi}(b) L(s, \psi)\,,
\end{aligned}
$$

where ψ_o is the principal character mod q. If $\chi(-1) = 1$, then we have

$$
\sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \chi(b) \bar{\psi}_o(b) = \frac{1}{q} \bar{\chi}(p) \sum_{j=1}^{q-1} j \chi(j) = 0\,.
$$

Thus when $\chi(-1) = 1$, we have

$$
Z_{\frac{p}{q}}(1, \chi) = \frac{1}{\varphi(q)} \sum_{\psi \neq \psi_o \bmod q} \left(\sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \chi(b) \bar{\psi}(b) \right) L(1, \psi)\,.
$$

Since for $\psi(-1) = -1$

$$
\sum_{b=1}^{q-1} \left(\left\{ \frac{p}{q} b \right\} - \frac{1}{2} \right) \chi(b) \bar{\psi}(b) = (\bar{\chi}\psi)(p) \frac{1}{q} \sum_{j=1}^{q-1} j (\chi \bar{\psi})(j) \neq 0\,,
$$

we get

$$
Z_{\frac{p}{q}}(1, \chi) = \frac{1}{\varphi(q) q} \sum_{\substack{\psi \neq \psi_o \bmod q \\ \psi(-1) = -1}} (\bar{\chi}\psi)(p) \sum_{j=1}^{q-1} j (\chi \bar{\psi})(j)) L(1, \psi)\,.
$$

We suppose further that $(q, \varphi(q)) = 1$. Thus we can apply Lemma C again and conclude that

$$
\sum_{n=1}^{\infty} \frac{\left\{ \frac{p}{q} n \right\} - \frac{1}{2}}{n} \chi(n)
$$

is transcendental.

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